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More on perfectly normal non-realcompact spaces

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Abstract

We use the space $(\omega_1, \tau(\vec{C}))$ associated with a guessing sequence $\vec{C}$ on ω_1 to show that it is consistent with CH that there exists a locally countable, first-countable, locally compact, perfectly normal, non-realcompact space of size $\aleph_1$ which does not contain any sub-Ostaszewski spaces. By a similar technique, it is shown to be consistent with MA $+\neg$CH that there exists a locally countable, first-countable, perfectly normal, non-realcompact space of size $\aleph_1$.

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0. Introduction

We are motivated by the following question, which has been open since it was asked by Blair as early as 1962.

Question 1. (*Blair.*) Does there always exist a perfectly normal, non-realcompact space under ZFC?

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Although the existence of such spaces is known to be consistent, very few examples have been constructed. The easiest one is the discrete space of size bigger than or equal to the first measurable cardinal. In [1], Ostaszewski built another example assuming $\clubsuit$, which is a guessing principle proposed in the same paper. Since $\mathrm{MA}+\neg\mathrm{CH}$ refutes $\clubsuit$, it was conjectured that $\mathrm{MA}+\neg\mathrm{CH}$ implies that there is no perfectly normal, non-realcompact space of size $\aleph_1$. This conjecture was negatively solved by Hernández-Hernández and the author in [2]. However, their space does not satisfy nice properties satisfied by Ostaszewski's example, such as first-countability and local compactness. Thus we may wonder if there is a perfectly normal, non-realcompact space with these properties which is essentially different from Ostaszewski's example. This is the question that will be investigated in this paper.

Eklof, Mekler, and Shelah defined a topology $\tau(\vec{C})$ on ω_1 associated with a guessing sequence $\vec{C}$ on ω_1. This topology was effectively used in [2]. We shall first give a closer look at the relationship between the combinatorial properties of $\vec{C}$ and the topological properties of $(\omega_1, \tau(\vec{C}))$. As a result, simple combinatorial properties of $\vec{C}$ are shown to correspond to the regularity, first-countability, and local compactness of $(\omega_1, \tau(\vec{C}))$.

By using this relationship, we shall prove that

(i) it is consistent with CH that there exists a locally countable, first-countable, locally compact, perfectly normal, non-realcompact space of size $\aleph_1$ which does not contain any sub-Ostaszewski spaces, and
(ii) it is consistent with $\mathrm{MA}+2^{\aleph_0}=\aleph_2$ that there exists a locally countable, first-countable, perfectly normal, non-realcompact space of size $\aleph_1$.

The proofs of these results use the same idea as in [2]. Nonetheless, when $(\omega_1, \tau(\vec{C}))$ is first-countable, each C_γ has an unbounded set of successor ordinals as elements, and hence $\vec{C}$ cannot be a club guessing sequence, which was essential in the proof of Hernández-Hernández and the author. Instead, we shall force perfect normality of $(\omega_1, \tau(\vec{C}))$ directly. In case of (ii), careful investigation of the resulting model reveals that the desired properties of $(\omega_1, \tau(\vec{C}))$ are preserved by the standard poset to force $\mathrm{MA}+2^{\aleph_0}=\aleph_2$.

Balogh proved in [3] that $\mathrm{MA}+\neg\mathrm{CH}$ implies that every locally countable, locally compact, perfectly normal space of size $\aleph_1$ is realcompact. Thus it is impossible to require local compactness to the witness of (ii). This is interesting when we see how similar the proofs of (i) and (ii) are.

Since we know that $\mathrm{MA}+\neg\mathrm{CH}$ is not strong enough to kill all perfectly normal, non-realcompact spaces, we may ask if a stronger forcing axiom can kill such spaces. Although we have not solved this question yet, we shall prove a partial result that PFA implies that $(\omega_1, \tau(\vec{C}))$ is not perfectly normal and non-realcompact when each C_γ is closed in the order topology. We do not know if it is consistent with PFA that $(\omega_1, \tau(\vec{C}))$ can be perfectly normal and non-realcompact when C_γ does not have to be closed.

In the final section of this paper, we prove the consistency of the existence of a regular, Hausdorff, non-D-space X such that for every closed subspace Y of X, $e(Y)=L(Y)$. Again, X is of the form $(\omega_1, \tau(\vec{C}))$. It may exemplify how useful our construction is.

The structure of this paper is as follows. In Section 1, we go over the basic definitions and give a combinatorial condition of $\vec{C}$ which is equivalent to the regularity of $(\omega_1, \tau(\vec{C}))$.

In Section 2, we establish the equivalence between the combinatorial properties of $\vec{C}$ and the first-countability and local compactness of $(\omega_1, \tau(\vec{C}))$. These characterizations are used in later sections.

In Section 3, we construct a first-countable, locally countable, locally compact, perfectly normal, non-realcompact space of size $\aleph_1$. The strategy is the same as the one in [2]. However, if $(\omega_1, \tau(\vec{C}))$ is first-countable, by the result in Section 2, $\vec{C}$ is not a club guessing sequence. Thus instead of relying on the club guessing property, we need to directly deal with perfect normality.

In Section 4, we show that it is consistent with $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$ that there exists a first-countable, locally countable, perfectly normal, non-realcompact space of size $\aleph_1$. It answers the following question asked in [2]: is there a first-countable, perfectly normal, non-realcompact space under $\mathrm{MA} + \neg\mathrm{CH}$?

In Section 5, we prove that PFA implies that $(\omega_1, \tau(\vec{C}))$ is not a perfectly normal, non-realcompact space when each C_γ is closed in the order topology. This result gives a positive prospect to the conjecture that PFA implies that every perfectly normal space of size $\aleph_1$ is realcompact.

In Section 6, we shall show the consistency that there exists a regular, Hausdorff, non-D-space X such that for every closed subspace Y of X, $e(Y) = L(Y)$. We again construct a guessing sequence $\vec{C}$ such that $(\omega_1, \tau(\vec{C}))$ has these properties.

1. Definition and basic facts

Most of our notations are standard. Lim stands for the class of all limit ordinals. When X and Y are sets of ordinals, we say that X is almost contained in Y and denote $X \subseteq^* Y$ if and only if there exists a $\zeta < \sup X$ such that $X \setminus \zeta \subseteq Y$. We also say that X and Y are almost equal, denoted by $X =^* Y$, if and only if $X \subseteq^* Y$ and $Y \subseteq^* X$. We use interval notations of ordinals. For example, $(\zeta, \delta]$ means $\{\gamma\colon \zeta < \gamma \leqslant \delta\}$. $\triangle$ denotes the well-ordering on the universe of a structure.

Realcompactness was introduced by Hewitt and Nachbin. There are several equivalent definitions of this property. We shall use the following one throughout this paper.

Definition 1.1. Let X be a topological space. We say that a subset Y of X is a *zero-set* if and only if there exists a real-valued continuous function f on X such that $Y = f^{\leftarrow}\{0\}$. A *z-filter* is a filter consisting of zero-sets. A maximal z-filter is called a *z-ultrafilter*.

We say that a filter $\mathcal{F}$ is fixed if and only if there is a singleton belonging to $\mathcal{F}$. Otherwise, $\mathcal{F}$ is said to be free.

We say that X is *realcompact* if and only if every z-ultrafilter with countable intersection property is fixed.

We shall define guessing sequences on ω_1, which are repeatedly used in this paper.

Definition 1.2. We say that a sequence $\langle C_\gamma\colon \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ is a *guessing sequence* on ω_1 if and only if each C_γ is an unbounded subset of γ.

We often denote a guessing sequence by $\vec{C}$. Eklof, Mekler, and Shelah introduced the topology $\tau(\vec{C})$ associated with a guessing sequence $\vec{C}$. This topology was exploited by Hernández-Hernández and the author in [2]. The following definition is slightly different but equivalent to the original version.

Definition 1.3. Let $\vec{C} = \langle C_\gamma \colon \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ be a guessing sequence on ω_1. The topology $\tau(\vec{C})$ associated with $\vec{C}$ is defined by: Y is $\tau(\vec{C})$-open if and only if for every $\gamma \in Y \cap \mathrm{Lim}$, $C_\gamma \subseteq^* Y$.

It is easy to see that $(\omega_1, \tau(\vec{C}))$ is Hausdorff. Moreover, for every $\gamma < \omega_1$, $\gamma + 1$ is a $\tau(\vec{C})$-closed set and γ is a $\tau(\vec{C})$-open set. Thus every uncountable subspace of $(\omega_1, \tau(\vec{C}))$ is not separable. In particular, since every sub-Ostaszewski space is hereditarily separable, $(\omega_1, \tau(\vec{C}))$ does not contain any sub-Ostaszewski spaces.

There is an easy equivalent condition for a subset of ω_1 to be $\tau(\vec{C})$-closed.

Lemma 1.4. *Let $\vec{C} = \langle C_\gamma \colon \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ be a guessing sequence on ω_1. For every subset F of ω_1, F is $\tau(\vec{C})$-closed if and only if for every $\gamma \in \omega_1 \cap \mathrm{Lim}$, whenever $C_\gamma \cap F$ is unbounded in γ, we have $\gamma \in F$.*

Proof. For every subset F of ω_1,

$$\begin{aligned}
&F \text{ is } \tau(\vec{C})\text{-closed}\\
&\quad\Longleftrightarrow \omega_1 \setminus F \text{ is } \tau(\vec{C})\text{-open}\\
&\quad\Longleftrightarrow \forall \gamma \in (\omega_1 \setminus F) \cap \mathrm{Lim}\ (C_\gamma \subseteq^* \omega_1 \setminus F)\\
&\quad\Longleftrightarrow \forall \gamma \in \omega_1 \cap \mathrm{Lim}\ (C_\gamma \not\subseteq^* \omega_1 \setminus F \Longrightarrow \gamma \in F)\\
&\quad\Longleftrightarrow \forall \gamma \in \omega_1 \cap \mathrm{Lim}\ (C_\gamma \cap F \text{ is unbounded in } \gamma \Longrightarrow \gamma \in F). \qquad \square
\end{aligned}$$

The necessary and sufficient condition for $(\omega_1, \tau(\vec{C}))$ to be regular is given as follows.

Lemma 1.5. *Let $\vec{C} = \langle C_\gamma \colon \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ be a guessing sequence on ω_1. The following are equivalent*:

(i) $(\omega_1, \tau(\vec{C}))$ *is regular.*
(ii) *For each $\gamma \in \omega_1 \cap \mathrm{Lim}$, there exists a $\zeta_\gamma < \gamma$ such that $\{\gamma\} \cup (C_\gamma \setminus \zeta_\gamma)$ is $\tau(\vec{C})$-closed.*

Proof. Let $\tau = \tau(\vec{C})$. (ii)⇒(i) is essentially proved in [2], but we shall present the proof here for reader's convenience. Assume (ii). First, we shall show that $(\delta, \tau \restriction \delta)$ is regular for every $\delta < \omega_1$ by induction. Suppose that for every $\gamma < \delta$, $(\gamma, \tau \restriction \gamma)$ is regular. We shall show that $(\delta, \tau \restriction \delta)$ is regular. Let $\gamma \in \delta$ and F a $(\tau \restriction \delta)$-closed set with $\gamma \notin F$. We shall build two pairwise disjoint $(\tau \restriction \delta)$-open sets U and W such that $\gamma \in U$ and $F \subseteq W$. First suppose that $\gamma + 1 \neq \delta$. Then by inductive hypothesis, $(\gamma + 1, \tau \restriction (\gamma + 1))$ is regular. Thus there exist two pairwise disjoint $(\tau \restriction (\gamma + 1))$-open sets U and $\overline{W}$ such that $\gamma \in U$ and $F \cap (\gamma + 1) \subseteq \overline{W}$. Let $W = \overline{W} \cup (\gamma, \delta)$. It is easy to see that W is $(\tau \restriction \delta)$-open and disjoint from U.

Now suppose that $\gamma+1=\delta$. If γ is a successor ordinal, let $U=\{\gamma\}$ and $W=\gamma$. Clearly it works. Suppose that γ is a limit ordinal. Since $(\gamma, \tau\restriction\gamma)$ is regular and countable, it is normal. By assumption, $\{\gamma\}\cup(C_\gamma\setminus\zeta_\gamma)$ is $(\tau\restriction\gamma)$-closed. Since F is $(\tau\restriction\delta)$-closed and $\gamma\notin F$, there exists a $\zeta<\gamma$ such that $\zeta\geqslant\zeta_\gamma$ and $(C_\gamma\setminus\zeta)\cap F=\emptyset$. Thus there exist two pairwise disjoint $(\tau\restriction\gamma)$-open sets $\overline{U}$ and W such that $C_\gamma\setminus\zeta\subseteq\overline{U}$ and $F\subseteq W$. Let $U=\overline{U}\cup\{\gamma\}$. It is easy to see that U is $(\tau\restriction\delta)$-open and disjoint from W.

Now let $\gamma<\omega_1$ and F a τ-closed set with $\gamma\notin F$. Since $(\gamma+1, \tau\restriction(\gamma+1))$ is regular, there exist two pairwise disjoint $\tau\restriction(\gamma+1)$-open sets U and $\overline{W}$ such that $\gamma\in U$ and $F\cap(\gamma+1)\subseteq\overline{W}$. Let $W=\overline{W}\cup(\gamma,\omega_1)$. As above, we can show that U and W separate γ and F.

For (i)⇒(ii), suppose that (ω_1,τ) is regular and let γ be the least such that there is no $\zeta<\gamma$ such that $\{\gamma\}\cup(C_\gamma\setminus\zeta)$ is τ-closed. Then $(\mathrm{cl}_\tau(C_\gamma)\setminus C_\gamma)\cap\gamma$ is unbounded in γ. Let $\langle\xi_n\colon n<\omega\rangle$ be an increasing cofinal sequence in $(\mathrm{cl}_\tau(C_\gamma)\setminus C_\gamma)\cap\gamma$. Then $\{\xi_n\colon n<\omega\}$ is τ-closed. Since (ω_1,τ) is regular, there exist two disjoint τ-open sets U and W such that $\gamma\in U$ and $\{\xi_n\colon n<\omega\}\subseteq W$. Since U is τ-open and $\gamma\in U$, there exists a $\zeta<\gamma$ such that $C_\gamma\setminus\zeta\subseteq U$. Let $n<\omega$ be such that $\xi_n>\zeta$. Since $\xi_n\in\mathrm{cl}_\tau(C_\gamma)$, there exists an $\eta\in(W\setminus(\zeta+1))\cap C_\gamma$. But since $C_\gamma\setminus\zeta\subseteq U$, we have $\eta\in U$. It contradicts that U and W are disjoint. □

All guessing sequences we use in this paper satisfy the condition of the previous lemma and hence the spaces associated with the sequences are regular. Since every regular, locally countable space is zero-dimensional, the spaces are also zero-dimensional.

From the next section, we shall discuss how to add more properties to the space.

2. First-countability and locally compactness

In this section, we shall establish some combinatorial properties of a guessing sequence which are equivalent to properties of the topological space associated with the sequence. These lemmas will be used later.

Lemma 2.1. *Let $\vec{C}=\langle C_\gamma\colon \gamma\in\omega_1\cap\mathrm{Lim}\rangle$ be a guessing sequence such that $(\omega_1,\tau(\vec{C}))$ is regular. Then the following are equivalent*:

(i) *$(\omega_1,\tau(\vec{C}))$ is first-countable.*
(ii) *For every $\delta\in\omega_1\cap\mathrm{Lim}$, there exists a $\zeta_\delta<\omega_1$ such that if $\gamma\in(C_\delta\setminus\zeta_\delta)\cap\mathrm{Lim}$, then $C_\gamma\subseteq^* C_\delta$.*

Proof. Let $\tau=\tau(\vec{C})$.

Suppose that (ii) holds. By induction on $\delta\in\omega_1\cap\mathrm{Lim}$, we shall show that $\{\{\delta\}\cup(C_\delta\setminus\zeta)\colon \zeta_\delta<\zeta<\delta \text{ and } \zeta\notin\mathrm{Lim}\}$ is a τ-open neighborhood base of δ. By the definition of τ, it suffices to show that $\{\delta\}\cup(C_\delta\setminus\zeta)$ is τ-open for every $\zeta\in(\zeta_\delta,\delta)\setminus\mathrm{Lim}$. Let $\gamma\in(C_\delta\setminus\zeta)\cap\mathrm{Lim}$. Note that since ζ is not a limit ordinal, we have $\zeta<\gamma$. Since $C_\gamma\subseteq^* C_\delta$, we have $C_\gamma\setminus\varepsilon\subseteq C_\delta$ for some $\varepsilon<\gamma$. Without loss of generality, we may assume that $\varepsilon\geqslant\zeta$. Then $C_\gamma\setminus\varepsilon\subseteq C_\delta\setminus\zeta$. Therefore $C_\delta\setminus\zeta$ is τ-open.

Conversely suppose (i) holds. Let $\delta \in \omega_1 \cap \mathrm{Lim}$. Then there exists a countable τ-open neighborhood base $\{N_i\colon i < \omega\}$ of δ. Suppose that (ii) fails at δ, i.e. there exist unboundedly many $\gamma \in C_\delta \cap \mathrm{Lim}$ such that $C_\gamma \nsubseteq^* C_\delta$. For each $i < \omega$, we define ξ_i as follows. Since N_i is a τ-open neighborhood of δ, there exists a $\zeta_i < \delta$ such that $C_\delta \setminus \zeta_i \subseteq N_i$. By assumption, there exists a $\gamma_i \in C_\delta \setminus \zeta_i$ such that $C_{\gamma_i} \nsubseteq^* C_\delta$. Note that since $\gamma_i \in N_i$, there exists an $\varepsilon_i < \gamma_i$ such that $C_{\gamma_i} \setminus \varepsilon_i \subseteq^* N_i$. Since $C_{\gamma_i} \nsubseteq^* C_\delta$, there exists a $\xi_i \in C_{\gamma_i} \setminus C_\delta$ such that $\xi_i \geqslant \varepsilon_i$. Then we have $\xi_i \in N_i$.

Now consider $X = \{\xi_i\colon i < \omega\}$. Since $\xi_i \notin C_\delta$ for every $i < \omega$, δ is not a τ-limit point of X. However, X meets every N_i and hence δ is a τ-limit point of X. This is a contradiction. $\square$

The following lemma is well known.

Lemma 2.2. *If X is a locally countable, locally compact topological space, then X is first-countable.*

The combinatorial condition of $\vec{C}$ which implies $(\omega_1, \tau(\vec{C}))$ is locally compact is given in the following lemma.

Lemma 2.3. *Let $\vec{C} = \langle C_\gamma\colon \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ be a guessing sequence such that $(\omega_1, \tau(\vec{C}))$ is regular. Then the following are equivalent*:

(i) *$(\omega_1, \tau(\vec{C}))$ is locally compact.*
(ii) *For every $\delta \in \omega_1 \cap \mathrm{Lim}$, there exists a $\zeta_\delta < \delta$ such that*
- *$C_\delta \setminus \zeta_\delta$ is closed in the order topology and*
- *for every $\gamma \in (C_\delta \setminus \zeta_\delta) \cap \mathrm{Lim}$, $C_\gamma =^* C_\delta \cap \gamma$.*

Proof. Let $\tau = \tau(\vec{C})$.

Suppose (ii). Fix $\delta < \omega_1$ and a τ-open set U with $\delta \in U$. We shall construct a τ-open neighborhood of δ which is compact and contained in U. If δ is 0 or successor, $\{\delta\}$ is τ-open and compact. Suppose that δ is a limit ordinal. By assumption, there exists a $\zeta_\delta < \delta$ such that $C_\delta \setminus \zeta_\delta$ is closed in the order topology and for every $\gamma \in (C_\delta \setminus \zeta_\delta) \cap \mathrm{Lim}$, $C_\gamma =^* C_\delta \cap \gamma$. We may assume that $C_\delta \setminus \zeta_\delta \subseteq U$. By the proof of Lemma 2.1, if ζ' is a successor ordinal with $\zeta_\delta < \zeta' < \delta$, then $\{\delta\} \cup (C_\delta \setminus \zeta')$ is τ-open. Let $N = \{\delta\} \cup (C_\delta \setminus (\zeta_\delta + 1))$. It suffices to show that N is τ-compact. Note that N is closed in the order topology.

Let $\{U_\alpha\colon \alpha < \kappa\}$ be a τ-open cover of N and assume that there is no finite subcover of $\{U_\alpha\colon \alpha < \kappa\}$. To derive a contradiction, we shall construct an infinite decreasing sequence $\langle \delta_n\colon n < \omega\rangle$ of ordinals in N by induction on n. We shall also define an $\alpha_n < \kappa$ at the nth stage so that $N \setminus \delta_n \subseteq \bigcup_{m \leqslant n} U_{\alpha_m}$. Let $\delta_0 = \delta$. Let $\alpha_0 < \kappa$ be such that $\delta_0 \in U_{\alpha_0}$. Let $\delta_1 = \sup(N \setminus U_{\alpha_0})$. Since N is closed in the order topology, we have $\delta_1 \in N$. We need to show that $\delta_1 < \delta_0$. Since U_{α_0} is τ-open, there exists an $\varepsilon_0 < \delta_0$ such that $\{\delta\} \cup (C_\delta \setminus \varepsilon_0) \subseteq U_{\alpha_0}$. Since $N = \{\delta\} \cup (C_\delta \setminus (\zeta_\delta + 1))$, we have $N \setminus U_{\alpha_0} \subseteq C_\delta \cap \varepsilon_0$. Thus we have $\delta_1 \leqslant \varepsilon_0 < \delta_0$.

Suppose that δ_m for $m \leqslant n$ and α_m for $m < n$ have been defined. Then since $\{U_\alpha\colon \alpha < \kappa\}$ covers N, there exists an α_n such that $\delta_n \in U_{\alpha_n}$. Let $\delta_{n+1} = \sup(N \setminus \bigcup_{m \leqslant n} U_{\alpha_m})$. Since N is closed in the order topology, we have $\delta_{n+1} \in N$. We shall show $\delta_{n+1} < \delta_n$. If δ_n is a successor ordinal, it is trivial. Suppose that δ_n is a limit ordinal. Since U_{α_n} is τ-open, there exists an $\varepsilon'_n < \delta_n$ such that $C_{\delta_n} \setminus \varepsilon'_n \subseteq U_{\alpha_n}$. Since $\delta_n \in N$ and $\delta_n < \delta$, we have $\delta_n \in C_\delta \setminus \zeta_\delta$. Thus $C_{\delta_n} =^* C_\delta \cap \delta_n$, i.e. there exists an $\varepsilon''_n < \delta_n$ such that $C_{\delta_n} \setminus \varepsilon''_n = (C_\delta \cap \delta_n) \setminus \varepsilon''_n$. Let $\varepsilon_n = \max\{\varepsilon'_n, \varepsilon''_n\}$. Then $(C_\delta \cap \delta_n) \setminus \varepsilon_n = C_{\delta_n} \setminus \varepsilon_n \subseteq U_{\alpha_n}$. Therefore we have

$$\begin{aligned}\left(N \setminus \bigcup_{m \leqslant n} U_{\alpha_m}\right) \setminus \varepsilon_n &\subseteq \big((N \cap \delta_n) \setminus U_{\alpha_n}\big) \setminus \varepsilon_n \\ &\subseteq \big((C_\delta \cap \delta_n) \setminus \varepsilon_n\big) \setminus U_{\alpha_n} \\ &\subseteq (C_{\delta_n} \setminus \varepsilon_n) \setminus U_{\alpha_n} \\ &= \emptyset.\end{aligned}$$

Hence $\delta_{n+1} \leqslant \varepsilon_n < \delta_n$. It finishes the inductive construction and we produced an infinite decreasing sequence, which is a contradiction.

Suppose (i). By Lemma 2.2, (ω_1, τ) is first-countable. By Lemma 2.1, for every $\delta \in \omega_1 \cap \mathrm{Lim}$, there exists a $\zeta < \delta$ such that for every $\gamma \in (C_\delta \setminus \zeta) \cap \mathrm{Lim}$, $C_\gamma \subseteq^* C_\delta$. Thus it suffices to show that for every $\delta \in \omega_1 \cap \mathrm{Lim}$, there exists a $\zeta_\delta < \delta$ such that $C_\delta \setminus \zeta_\delta$ is closed in the order topology and for every $\gamma \in (C_\delta \setminus \zeta_\delta) \cap \mathrm{Lim}$, $C_\delta \cap \gamma \subseteq^* C_\gamma$.

Fix $\delta \in \omega_1 \cap \mathrm{Lim}$. Let N be a τ-compact open neighborhood of δ. Then there exists an $\varepsilon < \delta$ such that $\{\delta\} \cup (C_\delta \setminus \varepsilon) \subseteq N$. Without loss of generality, we may assume that for every successor ordinal $\zeta \in (\varepsilon, \delta)$, $\{\delta\} \cup (C_\delta \setminus \zeta)$ is τ-open. First suppose that for every $\zeta < \delta$, $C_\delta \setminus \zeta$ is not closed in the order topology. In particular, $C_\delta \setminus \varepsilon$ is not closed in the order topology. Let γ be a limit point (in the order topology) of $C_\delta \setminus \varepsilon$ with $\gamma \notin C_\delta$. Let $\langle \xi_n \colon n < \omega \rangle$ be an increasing cofinal sequence in $(C_\delta \cap \gamma) \setminus \varepsilon$. Then $\{\xi_n \colon n < \omega\}$ is an infinite τ-closed discrete subset of N. It is a contradiction.

Now suppose that for every $\zeta < \delta$, there exists a $\gamma \in (C_\delta \setminus \zeta) \cap \mathrm{Lim}$ such that $C_\delta \cap \gamma \nsubseteq^* C_\gamma$. In particular, there exists such a $\gamma \in (C_\delta \setminus \varepsilon) \cap \mathrm{Lim}$. Let $\langle \xi_n \colon n < \omega \rangle$ be an increasing cofinal sequence in $(C_\delta \cap \gamma) \setminus (C_\gamma \cup \varepsilon)$. Then $\{\xi_n \colon n < \omega\}$ is an infinite τ-closed discrete subset of N. It is a contradiction. □

3. Construction of a first-countable, locally compact, perfectly normal, non-realcompact space

In this section, we shall prove the following theorem.

Theorem 3.1. *It is consistent with* CH *that there exists a locally countable, first-countable, locally compact, perfectly normal, non-realcompact space of size* $\aleph_1$ *such that the closure of every countable subset is countable. In particular, it does not contain any sub-Ostaszewski spaces.*

To this end, we build a guessing sequence $\langle C_\gamma \colon \gamma \in \omega_1 \cap \mathrm{Lim} \rangle$ via forcing such that $(\omega_1, \tau(\vec{C}))$ has these properties. The construction of the sequence is similar to the one used in [2]. To guarantee first countability and local compactness, we make sure that the equivalent conditions established in Lemmas 2.1 and 2.3 are satisfied. However, this makes it impossible for $\vec{C}$ to be a club guessing sequence, whose strengthening is the key in [2]. Instead of relying on guessing properties, we deal with perfect normality more directly. We also arrange that every stationary $\tau(\vec{C})$-closed set contains a club subset of ω_1, which implies that $(\omega_1, \tau(\vec{C}))$ is not realcompact. Moreover, as we observed, the closure of every countable subset is countable and hence $(\omega_1, \tau(\vec{C}))$ does not contain any sub-Ostaszewski spaces.

First recall the following definitions.

Definition 3.2. A poset P is said to be totally proper if and only if it is proper and adds no new reals.

Suppose that λ is a sufficiently large regular cardinal, and M is a countable elementary substructure of $\langle H(\lambda), \in, \triangle\rangle$ with $P \in M$. We say that a condition $p \in P$ is totally (M, P)-generic if and only if p is (M, P)-generic and p belongs to every open dense subset of P lying in M.

Clearly, P is totally proper if and only if for every sufficiently large regular cardinal λ, countable elementary substructure M of $\langle H(\lambda), \in, \triangle\rangle$ with $P \in M$, and $p \in P \cap M$, there exists a $q \leqslant p$ which is totally (M, P)-generic. In order to find a totally (M, P)-generic condition, we often use the following notion.

Definition 3.3. Let M and P be as in the previous definition. We say that a decreasing sequence $\langle p_n \colon n < \omega\rangle$ in P is an (M, P)-generic sequence if and only if each p_n belongs to M and for every open dense subset $\mathcal{D}$ of P, there exists an $n < \omega$ such that $p_n \in \mathcal{D}$.

For every $p \in P \cap M$, we can easily construct an (M, P)-generic sequence $\langle p_n \colon n < \omega\rangle$ with $p_0 = p$. It is easy to see that P is totally proper if and only if for every sufficiently large regular cardinal λ, countable elementary substructure M of $\langle H(\lambda), \in, \triangle\rangle$ with $P \in M$, and $p \in P \cap M$, there exists an (M, P)-generic sequence $\langle p_n \colon n < \omega\rangle$ with $p_0 = p$ which has a lower bound.

The following lemmas will help us prove perfect normality of $(\omega_1, \tau(\vec{C}))$.

Lemma 3.4. *Let D be a club subset of ω_1 and $X \subseteq \omega_1$. Then there exists an increasing sequence $\langle X_n \colon n < \omega\rangle$ of subsets of X such that $\bigcup_{n<\omega} X_n = X$ and for every $n < \omega$, all limit points $\gamma < \omega_1$ of X_n belong to D.*

Proof. Let $\langle \gamma_i \colon i < \omega_1\rangle$ be an increasing enumeration of D. Define $I_0 = [0, \gamma_0)$ and $I_{1+i} = [\gamma_i, \gamma_{i+1})$ for every $i < \omega_1$. For every $i < \omega_1$, let $f_i : I_i \cap X \to \omega$ be an injection. Define $X_n = \{\gamma \in X \colon f_i(\gamma) \leqslant n$ for some $i < \omega_1\}$. Then clearly $\langle X_n \colon n < \omega\rangle$ satisfies the conclusion. □

Lemma 3.5. *Let $\vec{C}$ be a guessing sequence such that $(\omega_1, \tau(\vec{C}))$ is regular. If F and H are pairwise disjoint non-stationary $\tau(\vec{C})$-closed sets, then there exist pairwise disjoint $\tau(\vec{C})$-open sets U_0 and U_1 with $F \subseteq U_0$ and $H \subseteq U_1$.*

Proof. Let $\tau = \tau(\vec{C})$. Let D be a club subset of ω_1 which is disjoint from both F and H. Let ξ_i be the $(1+i)$th element of D for every $i < \omega_1$. Let $I_0 = [0, \xi_0)$ and for each $i < \omega_1$, let $I_{1+i} = (\xi_i, \xi_{i+1})$. Note that $\bigcup_{i<\omega_1} I_i = \omega_1 \setminus D$ and hence F and H are contained in $\bigcup_{i<\omega_1} I_i$. Moreover, $(I_i, \tau \upharpoonright I_i)$ is countable and regular which implies that it is normal. Thus there exist two pairwise disjoint τ-open subsets W_i^0 and W_i^1 of I_i such that $F \cap I_i \subseteq W_i^0$ and $H \cap I_i \subseteq W_i^1$ for every $i < \omega_1$. Let $U_0 = \bigcup_{i<\omega_1} W_i^0$ and $U_1 = \bigcup_{i<\omega_1} W_i^1$. Then it is easy to see that U_0 and U_1 are as desired. $\square$

Lemma 3.6. *Let $\vec{C}$ be a guessing sequence on ω_1. Suppose that $(\omega_1, \tau(\vec{C}))$ is normal and every stationary $\tau(\vec{C})$-closed set contains a club subset of ω_1. Then $(\omega_1, \tau(\vec{C}))$ is perfectly normal if and only if for every club subset D of ω_1, there exists a club subset E of ω_1 such that $E \subseteq D$ and E is $\tau(\vec{C})$-G_δ.*

Proof. Let $\tau = \tau(\vec{C})$. If (ω_1, τ) is perfectly normal, then clearly every club subset of ω_1 is τ-G_δ. Suppose that every stationary τ-closed set contains a club subset of ω_1 and every club subset of ω_1 has a club subset which is τ-G_δ. Let F be a τ-closed set and we shall show that F is τ-G_δ. There are 2 cases.

Case 1. F is non-stationary.

Let D be a club subset of ω_1 which is disjoint from F. By Lemma 3.4, there exists an increasing sequence $\langle F_n \colon n < \omega \rangle$ of sets such that $\bigcup_{n<\omega} F_n = \omega_1 \setminus F$ and for every $n < \omega$, every limit point of F_n in the order topology belongs to D. Without loss of generality, we may assume that $D \subseteq F_n$. Thus it suffices to show that F_n is τ-closed for every $n < \omega$. Let δ be a τ-limit point of F_n. Since τ is finer than the order topology, δ is a limit point of F_n in the order topology. By assumption, it follows that $\delta \in D \subseteq F_n$. Therefore F_n is τ-closed.

Case 2. F is stationary.

By assumption, there exists a club subset D of ω_1 such that $D \subseteq F$. Then there exists a club subset E contained in D such that E is τ-G_δ, i.e. there exists an increasing sequence $\langle F_n \colon n < \omega \rangle$ of τ-closed subsets such that $\bigcup_{n<\omega} F_n = \omega_1 \setminus E$. By Lemma 3.4, we can pick an increasing sequence $\langle H_n \colon n < \omega \rangle$ such that $\bigcup_{n<\omega} H_n = \omega_1 \setminus F$ and for every $n < \omega$, every limit point of H_n in the order topology belongs to E. Then we have $\bigcup_{n<\omega} (F_n \cap H_n) = \omega_1 \setminus F$. It suffices to show that $F_n \cap H_n$ is τ-closed for every $n < \omega$. Let δ be a τ-limit point of $F_n \cap H_n$. By the same argument as above, we can show $\delta \in E$. However, we have $\delta \in \mathrm{cl}_\tau(F_n \cap H_n) \cap E \subseteq \mathrm{cl}_\tau(F_n) \cap E = F_n \cap E = \emptyset$. It is a contradiction and hence there is no τ-limit point of $F_n \cap H_n$. Therefore $F_n \cap H_n$ is τ-closed. $\square$

Lemma 3.7. *Let $\vec{C} = \langle C_\gamma \colon \gamma \in \omega_1 \cap \mathrm{Lim} \rangle$ be a guessing sequence such that $(\omega_1, \tau(\vec{C}))$ is regular. Suppose that F and H are disjoint $\tau(\vec{C})$-closed sets, F contains a club subset D of ω_1, and D and H are separated by disjoint $\tau(\vec{C})$-open sets. Then F and H are also separated by disjoint $\tau(\vec{C})$-open sets.*

Proof. Let $\tau = \tau(\vec{C})$. Let F, H, and D be as in the assumption. Then there exist pairwise disjoint τ-open sets U_1 and U_2 such that $D \subseteq U_1$ and $H \subseteq U_2$. By Lemma 3.5, there exist pairwise disjoint τ-open sets W_1 and W_2 such that $F \setminus D \subseteq W_1$ and $H \subseteq W_2$. Then clearly $U_1 \cup W_1$ and $U_2 \cap W_2$ separate F and H. $\square$

Now we shall define the countable-support iteration $\langle P_\alpha, \dot{Q}_\beta\colon \alpha \leqslant \omega_2,\ \beta < \omega_2\rangle$. During the construction, we shall also define a name $\dot{I}$ for a subset of ω_2 such that $\dot{I} \cap (\alpha + 1)$ is essentially a P_α-name for every $\alpha < \omega_2$. Let I be the interpretation of $\dot{I}$ in an appropriate extension. Assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Let Q_0 be defined by: $q \in Q_0$ if and only if there exists a $\delta < \omega_1$ (called the height of q and denoted by $\mathrm{ht}(q)$) such that q is a function from $(\delta + 1) \cap \mathrm{Lim}$ such that for every $\gamma \in (\delta + 1) \cap \mathrm{Lim}$,

(i) $q(\gamma)$ is an unbounded subset of γ,
(ii) $q(\gamma)$ is closed in the order topology in γ, and
(iii) if $\xi \in q(\gamma) \cap \mathrm{Lim}$, then $q(\xi) =^* q(\gamma) \cap \gamma$.

Q_0 is ordered by extension. If $G \subseteq Q_0$ is generic, then let $C_\gamma = q(\gamma)$ for some (all) $q \in G$ with $\gamma \leqslant \mathrm{ht}(q)$ for every $\gamma \in \omega_1 \cap \mathrm{Lim}$. Set $\vec{C} = \langle C_\gamma\colon \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ and $\tau(\vec{C}) = \tau$. By Lemmas 2.1 and 2.3, (ω_1, τ) is first-countable and locally compact. Let $\dot{\tau}$ be a P_1-name for τ and $\dot{C}_\gamma$ a P_1-name for C_γ for each $\gamma \in \omega_1 \cap \mathrm{Lim}$. Let $0 \notin I$.

Let $\langle \dot{F}_\alpha\colon 0 < \alpha < \omega_2\rangle$ be a bookkeeping of all τ-closed sets. If we are in an appropriate extension, let F_α be the interpretation of $\dot{F}_\alpha$.

Suppose that P_α has been defined and is totally proper. Let $G_\alpha \subseteq P_\alpha$ be generic and work in $V[G_\alpha]$. We define a poset Q_α as follows. If F_α is non-stationary, let Q_α be the trivial poset and $\alpha \notin I$. Suppose that F_α is stationary. In this case, let $\alpha \in I$. We define Q_α to be the set of all q such that there exists a $\delta < \omega_1$ such that

(i) $q : \delta + 1 \to \omega + 1$,
(ii) $q^{\leftarrow}\{\omega\} \subseteq F_\alpha \cap \mathrm{Lim}$,
(iii) $q^{\leftarrow}\{\omega\}$ is closed in the order topology in $\delta + 1$,
(iv) for every $n < \omega$, $q^{\leftarrow} n$ is τ-clopen, and
(v) $q(\delta) = \omega$.

We call this δ the height of q and denote by $\mathrm{ht}(q)$.

Let $\dot{Q}_\alpha$ be a name for Q_α. If $G_{\alpha+1} \subseteq P_{\alpha+1}$ is generic, define

$$D_\alpha = \bigl\{\gamma < \omega_1\colon\ p(\alpha)(\gamma) = \omega \text{ for some } p \in G_{\alpha+1}\bigr\},$$
$$H_{\alpha,n} = \bigl\{\gamma < \omega_1\colon\ p(\alpha)(\gamma) \leqslant n \text{ for some } p \in G_{\alpha+1}\bigr\}.$$

Clearly D_α is closed in the order topology, and $H_{\alpha,n}$ is τ-closed for every $n < \omega$. In case $\alpha \notin I$, we let $D_\alpha = \omega_1$ and $H_{\alpha,n} = \emptyset$ for all $n < \omega$. Let $\dot{D}_\alpha$ and $\dot{H}_{\alpha,n}$ be names for D_α and $H_{\alpha,n}$ respectively.

We need the following lemma to show that this construction works. The technique used in the proof was found by Foreman and Komjáth in [4] and applied by Hernández-Hernández and the author in [2].

Lemma 3.8. *For every $\alpha \leqslant \omega_2$, P_α is totally proper.*

Proof. Let λ be a sufficiently large regular cardinal. We go by induction on α. Clearly P_1 is totally proper.

Suppose that P_α is totally proper and we shall show that $P_{\alpha+1}$ is totally proper. First, we need to show the following claim.

Claim 1. *If $p \Vdash$ '$\alpha \in \dot{I}$' for some $p \in P_\alpha$, then $p \Vdash$ 'for every club subset D of ω_1, $\{q \in \dot{Q}_\alpha\colon \mathrm{ht}(q) \in D\}$ is dense'.*

Proof. Let $G_\alpha \subseteq P_\alpha$ be generic with $p \in G_\alpha$ and work in $V[G_\alpha]$. Let D be a club subset of ω_1 and $q \in Q_\alpha$. It suffices to build a $q' \leqslant q$ such that $\mathrm{ht}(q') \in D$. Pick a limit ordinal $\delta \in (D \cap F_\alpha) \setminus (\mathrm{ht}(q) + 1)$. Let $\langle \delta_n\colon n < \omega\rangle$ be an increasing cofinal sequence in δ with $\delta_0 = \mathrm{ht}(q)$. Define $q' \in P_\alpha$ by: $\mathrm{ht}(q') = \delta$ and for every $\gamma \in (\delta + 1) \cap \mathrm{Lim}$,

$$q'(\gamma) = \begin{cases} q(\gamma) & \text{if } \gamma \leqslant \mathrm{ht}(q), \\ n & \text{if } \delta_n < \gamma \leqslant \delta_{n+1}, \\ \omega & \text{if } \gamma = \delta. \end{cases}$$

Then it is easy to verify $q' \in Q_\alpha$. Clearly we have $q' \leqslant q$ and $\mathrm{ht}(q') = \delta \in D$. □

This claim implies that $\omega_1 \setminus D_\alpha = \bigcup_{n<\omega} H_{\alpha,n}$ in $V^{P_{\alpha+1}}$.

Claim 2. *$P_{\alpha+1}$ forces that for every finite subset B of $\dot{I}$, $\dot{D}_\alpha \cap \bigcap_{\beta\in B} \dot{D}_\beta$ is unbounded.*

Proof. Suppose that $p \in P_{\alpha+1}$ forces that B is a finite subset of $\dot{I}$ and let $\zeta < \omega_1$. Then clearly $p \restriction \alpha \Vdash$ '$\bigcap_{\beta\in B} \dot{D}_\beta$ is a club subset of ω_1'. By the previous claim, $p \restriction \alpha \Vdash$ 'there exists a $q \leqslant p(\alpha)$ such that $\mathrm{ht}(q) \in (\bigcap_{\beta\in B} \dot{D}_\beta) \setminus \zeta$'. Let $p' \in P_{\alpha+1}$ be defined by $p' \restriction \alpha = p \restriction \alpha$ and $p' \restriction \alpha \Vdash$ '$\mathrm{ht}(p'(\alpha)) \in (\bigcap_{\beta\in B} \dot{D}_\beta) \setminus \zeta$'. It follows that $p' \Vdash$ '$\dot{D}_\alpha \cap \bigcap_{\beta\in B} \dot{D}_\beta$ is not bounded by ζ'. Therefore, $P_{\alpha+1}$ forces that $\dot{D}_\alpha \cap \bigcap_{\beta\in B} \dot{D}_\beta$ is unbounded. □

Note that once we show that $P_{\alpha+1}$ preserves $\aleph_1$, the previous claim is trivial. However, it is required to show that $P_{\alpha+1}$ indeed preserves $\aleph_1$.

Claim 3. *$P_{\alpha+1}$ is totally proper.*

Proof. Let M be a countable elementary substructure of $\langle H(\lambda), \in, \triangle\rangle$ with $P_{\alpha+1}, \dot{F}_\alpha \in M$, $\delta = M \cap \omega_1$, and $p \in P_{\alpha+1} \cap M$. We shall show that there exists a totally $(M, P_{\alpha+1})$-generic condition $p' \leqslant p$. If $p \nVdash$ '$\alpha \in \dot{I}$', then there exists a $p_1 \leqslant p$ with $p_1 \in M$ such that $p_1 \Vdash$ '$\alpha \notin \dot{I}$ and hence $\dot{Q}_\alpha$ is trivial'. Thus by total properness of P_α, there exists a totally $(M, P_{\alpha+1})$-generic condition $p' \leqslant p_1$.

Suppose that $p \Vdash$ '$\alpha \in \dot{I}$'. Let $\langle p_i\colon i < \omega\rangle$ be an $(M, P_{\alpha+1})$-generic sequence with $p_0 = p$. We shall build sequences $\langle \gamma_n\colon n < \omega\rangle$ and $\langle W_n\colon n < \omega\rangle$ such that for every $n < \omega$, $p_i \Vdash$ 'W_n is a $\dot{\tau}$-clopen neighborhood of γ_n with $\gamma_n \in W_n \subseteq \gamma_n + 1$'. Let $\langle \delta_n\colon n < \omega\rangle$ be an increasing cofinal sequence in δ, $\bar{I} = \{\beta \in (0, \alpha)\colon \exists i < \omega(p_i \Vdash$ '$\beta \in \dot{I}$'$)\}$, and $\langle \beta_n\colon n < \omega\rangle$

an enumeration of $\bar{I}$. Suppose that we have defined γ_m and W_m for every $m < n$. Let $\delta'_n = \max\{\delta_n, \sup_{m<n} \gamma_m\}$. By the previous claim, there exists a $\gamma_n \in (\delta'_n, \delta)$ such that for some $i < \omega$, $p_i \Vdash$ '$\gamma_n \in \dot{D}_\alpha \cap \bigcap_{m \leqslant n} \dot{D}_{\beta_m}$'. Since $p \Vdash$ '$\dot{H}_{\alpha,n} \cup \bigcup_{m \leqslant n} \dot{H}_{\beta_m,n}$ is disjoint from $\dot{D}_\alpha \cap \bigcap_{m \leqslant n} \dot{D}_{\beta_m}$', we have $p_i \Vdash$ '$\gamma_n \notin \dot{H}_{\alpha,n} \cup \bigcup_{m \leqslant n} \dot{H}_{\beta_m,n}$' for every $i < \omega$. If γ_n is a successor ordinal, then let $W_n = \{\gamma_n\}$. Suppose that γ_n is a limit ordinal. Since $p \Vdash$ '$\dot{H}_{\alpha,n} \cup \bigcup_{m \leqslant n} \dot{H}_{\beta_m,n}$ is $\dot{\tau}$-closed', there exists a successor ordinal $\zeta_n \in (\delta'_n, \gamma_n)$ such that for some $i < \omega$, $p_i \Vdash$ '$(\dot{C}_{\gamma_n} \setminus \zeta_n) \cap (\dot{H}_{\alpha,n} \cup \bigcup_{m \leqslant n} \dot{H}_{\beta_m,n}) = \emptyset$'. Let W_n be such that for some $i < \omega$, $p_i \Vdash$ '$W_n = \dot{C}_{\gamma_n} \setminus \zeta_n$'. In either way, for some $i < \omega$, $p_i \Vdash$ '$W_n \cap (\dot{H}_{\alpha,n} \cup \bigcup_{m \leqslant n} \dot{H}_{\beta_m,n}) = \emptyset$'.

Define $p' \in P_{\alpha+1}$ by induction as follows. First, let

$$p'(0) \restriction \delta = \bigcup_{i<\omega} p_i(0),$$
$$p'(0)(\delta) = \bigcup_{n<\omega} W_n.$$

We claim $p' \restriction 1 \in P_1$. $p'(0)(\delta)$ is clearly unbounded subset of δ. We shall show that $p'(0)(\delta)$ is closed in δ. Suppose that $\gamma < \delta$ is a limit point of $p'(0)(\delta)$. Let $n < \omega$ be the least such that $\gamma \leqslant \gamma_n$. If $\gamma = \gamma_n$, then $\gamma \in W_n \subseteq p'(0)(\delta)$. Suppose $\gamma < \gamma_n$. Then γ is a limit point of W_n. But since W_n is not a singleton, we have $p_i \Vdash$ '$W_n = \dot{C}_{\gamma_n} \setminus \zeta_n$' for some $i < \omega$. Hence W_n is closed in the order topology. Therefore, we have $\gamma \in W_n \subseteq p'(0)(\delta)$.

Let $\xi \in p'(0)(\delta) \cap \mathrm{Lim}$. We shall show that $p'(0)(\xi) =^* p'(0)(\delta) \cap \xi$. Let $n < \omega$ be the least such that $\xi \leqslant \gamma_n$. If $\xi = \gamma_n$, then it is clear from the definition of W_n. Suppose that $\xi < \gamma_n$. Then γ_n is a limit ordinal and for some $i < \omega$, $p_i \Vdash$ '$W_n = \dot{C}_{\gamma_n} \setminus \zeta_n$'. Clearly we have $\xi \geqslant \zeta_n$. Since ζ_n is a successor ordinal, we have $\xi > \zeta_n$. Then for all $i < \omega$ with $\gamma_n \leqslant \mathrm{ht}(p_i(0))$, $p'(0)(\xi) \setminus \zeta_n = p_i(0)(\xi) \setminus \zeta_n =^* (p_i(0)(\gamma_n) \setminus \zeta_n) \cap \xi = W_n \cap \xi = p'(0)(\delta) \cap [\zeta_n, \xi)$. Therefore, $p'(0)(\xi) =^* p'(0)(\delta) \cap \xi$.

Suppose that $p' \restriction \beta'$ has been defined for $\beta' \leqslant \beta$. If $\beta \notin \bar{I}$, then let $\beta \notin \mathrm{supp}(p')$. Otherwise, let $\beta \in \mathrm{supp}(p')$ and

$$p' \restriction \beta \Vdash \text{`}p'(\beta) = \bigcup_{i<\omega} p_i(\beta) \cup \{\langle \delta, \omega \rangle\}\text{'}.$$

We claim that $p' \restriction (\beta + 1) \in P_{\beta+1}$. If $\beta \notin \bar{I}$, then there is nothing to prove. Suppose that $\beta \in \bar{I}$. Then there exists an $m < \omega$ such that $\beta = \beta_m$. Since for some $i < \omega$, $p_i \Vdash$ '$\dot{F}_\beta \cap \dot{C}_\delta \supseteq \dot{D}_\beta \cap \dot{C}_\delta \supseteq \{\gamma_n\colon n \geqslant m\}$ is unbounded in δ and $\dot{F}_\beta$ is $\dot{\tau}$-closed', we have $p' \restriction \beta \Vdash$ '$\delta \in \dot{F}_\beta$'. We also have for every $m < n < \omega$, $p' \restriction \beta \Vdash$ '$(\dot{C}_\delta \setminus (\gamma_n + 1)) \cap \dot{H}_{\beta,n} = \bigcup_{n<k<\omega} W_k \cap \dot{H}_{\beta,n} = \emptyset$'. Hence $p' \restriction \beta \Vdash$ '$(p'(\beta))^{\leftarrow} n$ is $\dot{\tau}$-closed'. When $\beta = \alpha$, a similar argument works.

If β is a limit ordinal and $p' \restriction \beta' \in P_{\beta'}$ for every $\beta' < \beta$, then since $\mathrm{supp}(p' \restriction \beta) = \{0\} \cup (\bar{I} \cap \beta)$ is countable, we have $p' \restriction \beta \in P_\beta$. Thus we successfully built $p' \in P_{\alpha+1}$. It is easy to see that p' is totally $(M, P_{\alpha+1})$-generic. $\square$

Therefore, we can pass successor stages. A similar argument suffices for limit stages. $\square$

Let $P = P_{\omega_2}$. The following lemma proves the key properties satisfied in V^P.

Lemma 3.9. *Let $G \subseteq P$ be generic. Then in $V[G]$,*

(i) *every stationary τ-closed set contains a club subset of ω_1,*
(ii) *every club subset of ω_1 contains a club subset of ω_1 which is τ-G_δ, and*
(iii) *every non-stationary τ-closed set is contained in $\bigcup_{\beta\in B} H_{\beta,n} \cup \zeta$ for some finite subset B of I, $n < \omega$, and $\zeta < \omega_1$.*

Proof. (i) is immediate from the definition of P. To see (ii), let D be a club subset of ω_1. Then there exists an $\alpha \in I$ such that $D = F_\alpha$. Thus $D_\alpha \subseteq D$ is a club subset. Moreover, we have $D_\alpha = \omega_1 \setminus \bigcup_{n<\omega} H_{\alpha,n}$ and hence D_α is τ-G_δ.

For (iii), suppose that F is a τ-closed set which is not contained in any set of the form $\bigcup_{\beta\in B} H_{\beta,n} \cup \zeta$ for some finite subset B of I, $n < \omega$, and $\zeta < \omega_1$. We shall show that F is stationary. Let $\dot F$ be a P-name for F, $\dot D$ a P-name for a club subset of ω_1, and $p \in G$ a condition which forces the assumed properties of $\dot F$. Let M be a countable elementary substructure of $\langle H(\lambda), \in, \triangle\rangle$ with $P, p, \dot F, \dot D, \langle \dot H_{\alpha,n}\colon 0 < \alpha < \omega_2,\ n < \omega\rangle \in M$ and $\delta = M \cap \omega_1$. Let $\langle p_i\colon i < \omega\rangle$ be a (M, P)-generic sequence with $p_0 = p$. Let $\langle \delta_n\colon n < \omega\rangle$ be an increasing cofinal sequence in δ, $\bar I = \{\beta \in (0, \omega_2)\colon \exists i < \omega (p_i \Vdash \text{‘}\beta \in \dot I\text{’})\}$, and $\langle \beta_n\colon n < \omega\rangle$ an enumeration of $\bar I$. We shall build four sequences $\langle \gamma_n\colon n < \omega\rangle$, $\langle W_n\colon n < \omega\rangle$, $\langle \gamma'_n\colon n < \omega\rangle$, and $\langle W'_n\colon n < \omega\rangle$ as follows.

Suppose that we have defined γ_m, W_m, γ'_m, and W'_m for every $m < n$. Define $\delta'_n = \max\{\delta_n, \sup_{m<n} \gamma'_m\}$. By assumption, $p \Vdash \text{‘}\dot F \nsubseteq \bigcup_{m\leqslant n} \dot H_{\beta_m,n} \cup (\delta'_n + 1)\text{’}$. Thus there exists a $\gamma_n < \delta$ such that for some $i < \omega$, $p_i \Vdash \text{‘}\gamma_n \in \dot F \setminus (\bigcup_{m\leqslant n} \dot H_{\beta_m,n} \cup (\delta'_n + 1))\text{’}$. If γ_n is a successor ordinal, then let $W_n = \{\gamma_n\}$. Otherwise, there exists a successor ordinal $\zeta_n \in (\delta'_n, \gamma_n)$ such that for some $i < \omega$, $p_i \Vdash \text{‘}(\dot C_{\gamma_n} \setminus \zeta_n) \cap \bigcup_{m\leqslant n} \dot H_{\beta_m,n} = \emptyset\text{’}$. Let W_n be such that $p_i \Vdash \text{‘}W_n = \{\gamma_n\} \cup (\dot C_{\gamma_n} \setminus \zeta_n)\text{’}$. We clearly have $p \Vdash \text{‘}\bigcap_{m\leqslant n} \dot D_{\beta_m} \setminus (\gamma_n + 1)$ is club’. Then there exists a $\gamma'_n < \omega_1$ such that for some $i < \omega$, $p_i \Vdash \text{‘}\gamma'_n \in \bigcap_{m\leqslant n} \dot D_{\beta_m} \setminus (\gamma_n + 1)\text{’}$. If γ'_n is a successor ordinal, let $W'_n = \{\gamma'_n\}$. Otherwise, there exists a successor ordinal $\zeta'_n \in (\gamma_n, \gamma'_n)$ such that for some $i < \omega$, $p_i \Vdash \text{‘}(\dot C_{\gamma'_n} \setminus \zeta'_n) \cap \bigcup_{m\leqslant n} \dot H_{\beta_m,n} = \emptyset\text{’}$. Let W'_n be such that for some $i < \omega$, $p_i \Vdash \text{‘}W'_n = \{\gamma'_n\} \cup (\dot C_{\gamma'_n} \setminus \zeta'_n)\text{’}$.

We shall define $p' \in P$ by induction. Let

$$p'(0) \restriction \delta = \bigcup_{i<\omega} p_i(0),$$

$$p'(0)(\delta) = \bigcup_{n<\omega} (W_n \cup W'_n).$$

As in the previous argument, we can show $p' \restriction 1 \in P_1$.

Suppose that $p' \restriction \beta \in P_\beta$ has been defined. If $\beta \notin \bar I$, let $\beta \notin \operatorname{supp}(p')$. Otherwise, define

$$p' \restriction \beta \Vdash \text{‘}p'(\beta) = \bigcup_{i<\omega} p_i(\beta) \cup \{\langle \delta, \omega\rangle\}\text{’}.$$

As before, we can check that this inductive construction works and $p' \in P$. It is easily seen that $p' \Vdash$ ‘δ is a τ-limit point of $\dot{F}$ and hence $\delta \in \dot{F} \cap \dot{D}$’. Therefore, F is stationary in $V[G]$. $\square$

Let $G \subseteq P$ be generic. We shall show that (ω_1, τ) is as we desired. For normality, let F and H be disjoint τ-closed sets. By Lemma 3.5, we may assume that F is stationary. By Lemma 3.9(ii), F contains a club subset D of ω_1 and hence H is non-stationary. Then by Lemma 3.9(iii), there exist a finite subset B of I, an $n < \omega$, and a $\zeta < \omega_1$ such that $H \subseteq \bigcup_{\beta \in B} H_{\beta,n} \cup (\zeta + 1)$. Let $E = (D \cap \bigcap_{\beta \in B} D_\beta) \setminus (\zeta + 1)$, which is clearly club. Moreover, E and $\bigcup_{\beta \in B} H_{\beta,n} \cup (\zeta + 1)$ are disjoint. Since $\bigcup_{\beta \in B} H_{\beta,n} \cup (\zeta + 1)$ is τ-clopen, it means that E and H are separated by disjoint τ-open sets. By Lemma 3.7, we can conclude that F and H can be separated by disjoint τ-open sets.

Lemmas 3.6 and 3.9 imply that (ω_1, τ) is perfectly normal. Therefore, every closed set is a zero-set. By Lemma 3.9(i), the club filter restricted to τ-closed sets is a z-ultrafilter with countable intersection property. Thus (ω_1, τ) is not realcompact.

4. With MA + ¬CH

In [2], it is shown to be consistent with MA + ¬CH that there exists a perfectly normal, non-realcompact space of size $\aleph_1$. Extending this result, we shall prove the following theorem.

Theorem 4.1. *It is consistent with* MA + ¬CH *that there exists a locally countable, first-countable, perfectly normal, non-realcompact space of size* $\aleph_1$.

That is, we can additionally require first-countability. It answers the question asked in [2]. Assuming $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, we define a countable-support iteration of length ω_2 in a similar way as in the previous section. This iteration adds a guessing sequence $\vec{C}$ such that $(\omega_1, \tau(\vec{C}))$ is first-countable, perfectly normal, and non-realcompact. As in Section 3, instead of using a guessing property, we directly force perfect normality of $(\omega_1, \tau(\vec{C}))$. Then we force with the standard poset to force MA + ¬CH. Moreover, additional examination shows that $(\omega_1, \tau(\vec{C}))$ is already perfectly normal in the final model.

It is impossible to build a guessing sequence $\vec{C}$ such that $(\omega_1, \tau(\vec{C}))$ witnesses Theorem 4.1 and is also locally compact. The following theorem proved by Balogh in [3] refutes the possibility to add local compactness.

Theorem 4.2. (Balogh.) *If* MA + ¬CH *holds, then every locally countable, locally compact, perfectly normal space of size* $\aleph_1$ *is paracompact.*

Assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. We shall define a countable-support iteration $\langle P_\alpha, \dot{Q}_\beta\colon \alpha \leqslant \omega_2,\ \beta < \omega_2 \rangle$ and a name $\dot{I}$ almost in the same way as in the previous section. The only difference occurs at the 0th stage. Let Q_0 be defined by: $q \in Q_0$ if and only if there exists a $\delta < \omega_1$ (called the height of q and denoted by $\mathrm{ht}(q)$) such that q is a function from $(\delta + 1) \cap \mathrm{Lim}$ such that for every $\gamma \in (\delta + 1) \cap \mathrm{Lim}$,

(i) $q(\gamma)$ is an unbounded subset of γ,
(ii) for every $\xi \in \gamma \cap \mathrm{Lim}$, if $q(\xi) \cap q(\gamma)$ is unbounded in ξ, then $\xi \in q(\gamma)$, and
(iii) if $\xi \in q(\gamma) \cap \mathrm{Lim}$, then $q(\xi) \subseteq^* q(\gamma)$.

Q_0 is ordered by extension. If we define $\vec{C} = \langle C_\gamma\colon \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ in the same way as in the previous section, it is clear that $(\omega_1, \tau(\vec{C}))$ is first-countable, but not locally compact. Let $\dot{D}_\alpha$ and $\dot{H}_{\alpha,n}$ be defined as in the previous section.

We can show the following lemma by exactly the same method as for Lemma 3.8.

Lemma 4.3. *For every $\alpha \leqslant \omega_2$, P_α is totally proper.*

Let $P = P_{\omega_2}$. The following lemma asserts that in V^P, the same conclusions as in Lemma 3.9 hold even after ccc extension. We could not prove this lemma if we had used the same P as in the previous section. This is the reason why a similar argument cannot yields a result contradictory to Balogh's theorem.

Lemma 4.4. *Let $G \subseteq P$ be generic. In $V[G]$, let R be a ccc poset and take $G' \subseteq R$ to be generic. Then in $V[G][G']$,*

(i) *every stationary τ-closed set contains a club subset of ω_1,*
(ii) *every club subset of ω_1 contains a club subset of ω_1 which is τ-G_δ, and*
(iii) *every non-stationary τ-closed set is contained in $\bigcup_{\beta\in B} H_{\beta,n} \cup \zeta$ for some finite subset B of I, $n < \omega$, and $\zeta < \omega_1$.*

Before we begin the proof of this lemma, let us remark that we can prove the following lemma by the same argument as in Lemma 3.9.

Lemma 4.5. *Let $G \subseteq P$ be generic. Then in $V[G]$,*

(i) *every stationary τ-closed set contains a club subset of ω_1,*
(ii) *every club subset of ω_1 contains a club subset of ω_1 which is τ-G_δ, and*
(iii) *every non-stationary τ-closed set is contained in $\bigcup_{\beta\in B} H_{\beta,n} \cup \zeta$ for some finite subset B of I, $n < \omega$, and $\zeta < \omega_1$.*

Now we shall show that these properties are preserved by any ccc extension.

Proof of Lemma 4.4. Since every club subset of ω_1 in $V[G][G']$ contains a club subset of ω_1 in $V[G]$, (ii) is obvious.

In order to show (i) and (iii), we shall first prove the following claim. Let $\dot{R}$ be a P-names for R.

Claim 1. *Suppose that $p \in P$, $\dot{r}$ is a P-name for an element of $\dot{R}$, and $\dot{\dot{F}}$ is a P-name for an $\dot{R}$-name for a τ-closed set such that $\langle p, \dot{r}\rangle$ forces that there exist no finite subset B of I, $n < \omega$, and $\zeta < \omega_1$ such that $\dot{\dot{F}} \subseteq \bigcup_{\beta\in B} \dot{H}_{\beta,n} \cup \zeta$. Let M be a countable elementary* *substructure of* $\langle H(\lambda), \in, \triangle\rangle$ *with* $P, p, \dot{R}, \dot{r}, \dot{\dot{F}} \in M$ *and* $\delta = M \cap \omega_1$. *Then there exists a totally* (M, P)*-generic condition* $p' \leqslant p$ *such that* $p' \Vdash$ '$\dot{r} \Vdash$ 'δ *is a* τ*-limit point of* $\dot{\dot{F}}$''.

Proof. Let $\langle p_i \colon i < \omega\rangle$ be a (M, P)-generic sequence with $p_0 = p$. Let $\langle \delta_n \colon n < \omega\rangle$ be an increasing cofinal sequence in δ, $\bar{I} = \{\beta \in (0, \alpha) \colon \exists i < \omega (p_i \Vdash \text{'}\beta \in \dot{I}\text{'})\}$, and $\langle \beta_n \colon n < \omega\rangle$ an enumeration of $\bar{I}$.

We shall define a sequence $\langle W_n \colon n < \omega\rangle$ of bounded subsets of δ as follows. Suppose that we have defined W_m for all $m < n$. Let $\delta'_n = \max\{\delta_n, \sup \bigcup_{m<n} W_m\}$. Since $p \Vdash$ '$\bigcap_{m\leqslant n} \dot{D}_{\beta_m}$ is club', there exists a $\gamma_n < \delta$ such that for some $i < \omega$, $p_i \Vdash$ '$\gamma_n \in \bigcap_{m\leqslant n} \dot{D}_{\beta_m} \setminus (\delta'_n + 1)$'. Moreover, since $p \Vdash$ '$\dot{R}$ is ccc and $\dot{r} \Vdash$ '$\dot{\dot{F}} \setminus \bigcup_{m\leqslant n} \dot{H}_{\beta_m,n}$ is unbounded in ω_1'', there exists a $\gamma'_n < \delta$ such that $p \Vdash$ '$\dot{r} \Vdash$ '$(\dot{\dot{F}} \setminus \bigcup_{m\leqslant n} \dot{H}_{\beta_m,n}) \cap (\delta'_n, \gamma'_n] \neq \emptyset$''. Without loss of generality, we may assume $\gamma'_n \geqslant \gamma_n$. Let W_n be such that for some $i < \omega$, $p_i \Vdash$ '$W_n = (\delta'_n, \gamma'_n] \setminus \bigcup_{m\leqslant n} \dot{H}_{\beta_m,n}$'. Then clearly for some $i < \omega$, $p_i \Vdash$ 'W_n is a $\dot{\tau}$-clopen set such that $W_n \cap \bigcup_{m\leqslant n} \dot{H}_{\beta_m,n} = \emptyset$, $W_n \cap \bigcap_{m\leqslant n} \dot{D}_{\beta_m} \neq \emptyset$ and $\dot{r} \Vdash$ '$W_n \cap \dot{\dot{F}} \neq \emptyset$''.

Define $p' \in P$ by induction. Let

$$p'(0) \restriction \delta = \bigcup_{i<\omega} p_i(0),$$

$$p'(0)(\delta) = \bigcup_{n<\omega} W_n.$$

We claim that $p' \restriction 1 \in P_1$. It suffices to verify that (i)–(iii) of the definition of Q_0 are satisfied. (i) is clearly true. Since each W_n is forced to be $\dot{\tau}$-closed and $\sup W_n < \min W_{n+1}$ for every $n < \omega$, (ii) also holds. Since each W_n is forced to be $\dot{\tau}$-open, we can show (iii). Thus $p' \restriction 1 \in P_1$.

Suppose that $p' \restriction \beta$ has been defined for $\beta < \omega_2$. If $\beta \notin \bar{I}$, then let $\beta \notin \operatorname{supp}(p')$. Otherwise, let

$$p' \restriction \beta \Vdash \text{'}p'(\beta) = \bigcup_{i<\omega} p_i(\beta) \cup \{\langle \delta, \omega\rangle\}\text{'}.$$

We claim that $p' \restriction (\beta + 1) \in P_{\beta+1}$. Since $\beta \in \bar{I}$, there exists an $m < \omega$ such that $\beta = \beta_m$. Then for every $m < n < \omega$, $p' \restriction \beta \Vdash$ '$\dot{F}_\beta \cap W_n \supseteq \dot{D}_\beta \cap W_n \neq \emptyset$'. Hence $p' \restriction \beta \Vdash$ '$\dot{F}_\beta \cap \dot{C}_\delta$ is unbounded in δ and hence $\delta \in \dot{F}_\beta$'. Now it suffices to show that $p' \restriction \beta \Vdash$ 'δ is not a $\dot{\tau}$-limit point of $\dot{H}_{\beta,n}$'. But for every $m < n \leqslant k < \omega$, $p' \restriction \beta \Vdash$ '$W_k \cap \dot{H}_{\beta,n} = \emptyset$'. Thus $p' \restriction \beta \Vdash$ '$\dot{C}_\delta \cap \dot{H}_{\beta,n}$ is bounded in δ and hence δ is not a $\dot{\tau}$-limit point of $\dot{H}_{\beta,n}$'.

Since $\operatorname{supp}(p') = \{0\} \cup \bar{I}$ is countable, there is no problem at limit stages. Thus we can show $p' \in P$. Then it is easy to see that p' is totally (M, P)-generic. Moreover, we claim that $p' \Vdash$ '$\dot{r} \Vdash$ 'δ is a $\dot{\tau}$-limit point of $\dot{\dot{F}}$''. Let $\zeta < \delta$. Then there exists an $n < \omega$ such that $\delta_n > \zeta$. It follows that $\min(W_n) > \delta_n > \zeta$. But we have for some $i < \omega$, $p_i \Vdash$ '$\dot{r} \Vdash$ '$W_n \cap \dot{\dot{F}} \neq \emptyset$''. Therefore, $p' \Vdash$ '$\dot{r} \Vdash$ $(\dot{C}_\delta \setminus \zeta) \cap \dot{\dot{F}} \neq \emptyset$ and hence δ is a $\dot{\tau}$-limit point of $\dot{\dot{F}}$''. □

Claim 2. (iii) *holds.*

Proof. Let $p \in P$, $\dot{r}$ a P-name for an element of $\dot{R}$, and $\dot{\dot{F}}$ a P-name for an $\dot{R}$-name for a τ-closed set such that $p \Vdash$ 'for every finite subset B of $\dot{I}$, $n < \omega$, and $\zeta < \omega_1$, $\dot{r} \Vdash$ '$\dot{\dot{F}} \nsubseteq \bigcup_{\beta \in B} \dot{H}_{\beta,n} \cup \zeta$''. We shall show that $p' \Vdash$ '$\dot{r} \Vdash$ '$\dot{\dot{F}}$ is stationary''. Let $\dot{D}$ be a P-name for a club subset of ω_1 and let M be a countable elementary substructure of $\langle H(\lambda), \in, \triangle\rangle$ with $P, p, \dot{R}, \dot{r}, \dot{\dot{F}}, \dot{D} \in M$ and $\delta = M \cap \omega_1$. Then by Claim 1, there exists a totally (M, P)-generic $p' \leqslant p$ such that $p' \Vdash$ '$\dot{r} \Vdash$ 'δ is a $\dot{\tau}$-limit point of $\dot{\dot{F}}$''. Then $p' \Vdash$ '$\dot{r} \Vdash$ '$\delta \in \dot{D} \cap \dot{\dot{F}}$''. Therefore $p \Vdash$ '$\dot{r} \Vdash$ '$\dot{\dot{F}}$ is stationary''. □

Claim 3. (i) *holds.*

Proof. Let $p \in P$, $\dot{r}$ a P-name for an element of $\dot{R}$, and $\dot{\dot{F}}$ a P-name for an $\dot{R}$-name for a stationary $\dot{\tau}$-closed set. Suppose that $\dot{D}$ is a P-name for a club subset of ω_1 and let M be a countable elementary substructure of $\langle H(\lambda), \in, \triangle\rangle$ with $P, p, \dot{R}, \dot{r}, \dot{\dot{F}}, \dot{D} \in M$ and $\delta = M \cap \omega_1$. Notice that since $p \Vdash$ '$\dot{H}_{\alpha,n}$ is non-stationary for every $\alpha \in \dot{I}$ and $n < \omega$', we have $p \Vdash$ 'for every finite subset B of I, $n < \omega$, and $\zeta < \omega_1$, $\dot{r} \Vdash$ '$\dot{\dot{F}} \nsubseteq \bigcup_{\beta \in B} \dot{H}_{\beta,n} \cup \zeta$''. Then by Claim 1, there exists a totally (M, P)-generic $p' \leqslant p$ such that $p' \Vdash$ '$\dot{r} \Vdash$ 'δ is a $\dot{\tau}$-limit point of $\dot{\dot{F}}$''. Thus $p' \Vdash$ '$\dot{r} \Vdash$ '$\delta \in \dot{D} \cap \dot{\dot{F}}$''. Hence $p' \Vdash$ '$\{\gamma < \omega_1\colon \dot{r} \Vdash$ '$\gamma \in \dot{\dot{F}}$'$\}$ is stationary'. Therefore, in $V[G][G']$, every stationary τ-closed set contains a stationary τ-closed set in $V[G]$. But, by Lemma 4.5, every stationary τ-closed set in $V[G]$ contains a club subset of ω_1. Thus every stationary τ-closed set contains a club subset of ω_1. □

By Lemma 4.4, we can finish the proof of Theorem 4.1 as in the previous section.

5. PFA and $(\omega_1, \vec{C})$

As we have seen, there are many cases where $(\omega_1, \tau(\vec{C}))$ is perfectly normal and non-realcompact. Thus we may wonder if there always exists a guessing sequence $\vec{C}$ such that $(\omega_1, \tau(\vec{C}))$ is perfectly normal and non-realcompact, which answers Blair's question. Although we cannot completely reject this possibility, if we assume that each C_γ is closed in the order topology, PFA implies that $(\omega_1, \tau(\vec{C}))$ cannot be perfectly normal and non-realcompact. In this section, we shall prove this result.

We shall begin with the following easy observation.

Lemma 5.1. *Let $\vec{C} = \langle C_\gamma\colon \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$ be a guessing sequence. Suppose $(\omega_1, \tau(\vec{C})$ is a perfectly normal, non-realcompact space and $\mathcal{F}$ is a free z-ultrafilter with countable intersection property. Then the following statements hold*:

(i) *If $F \in \mathcal{F}$ is $\tau(\vec{C})$-closed, then* $\mathrm{der}_{\tau(\vec{C})}(F) \in \mathcal{F}$.
(ii) *If D is a club subset of ω_1, then $D \in \mathcal{F}$.*

Proof. Let $\tau = \tau(\vec{C})$.

(i) Suppose that F is τ-closed, $F \in \mathcal{F}$ and $\mathrm{der}_\tau(F) \notin \mathcal{F}$. Then there exists a τ-closed set $H \in \mathcal{F}$ such that $H \cap \mathrm{der}_\tau(F) = \emptyset$. Without loss of generality, we may assume that $H \subseteq F$.

But then clearly H is τ-closed discrete. Then $\mathcal{P}(H) \cap \mathcal{F}$ is a non-principal countably complete ultrafilter on H, which implies that $\aleph_1$ is measurable. It is a contradiction.

(ii) Let D be a club subset of ω_1 such that $D \notin \mathcal{F}$. Since $\mathcal{F}$ is a z-ultrafilter, there exists a τ-closed set $F \in \mathcal{F}$ such that $F \cap D = \emptyset$. By Lemma 3.4, there exists an increasing sequence $\langle F_n \colon n < \omega \rangle$ of subsets of F such that $\bigcup_{n<\omega} F_n = F$ and for every $n < \omega$, every limit point of F_n in the order topology belongs to D. Thus, for every $n < \omega$, every τ-limit point of F_n must belong to D. However, F is a τ-closed set disjoint from D, it follows that F_n is a τ-closed discrete set. By (i), we have $F_n \notin \mathcal{F}$ for every $n < \omega$. Hence for every $n < \omega$, there exists an $H_n \in \mathcal{F}$ disjoint from F_n. Then, we have $F \cap \bigcap_{n<\omega} H_n = \emptyset$. It is a contradiction since we assume that $\mathcal{F}$ has countable intersection property. $\square$

The following lemma holds without PFA.

Lemma 5.2. *Let $\vec{C} = \langle C_\gamma \colon \gamma \in \omega_1 \cap \mathrm{Lim} \rangle$ be a guessing sequence such that $(\omega_1, \tau(\vec{C}))$ is perfectly normal and non-realcompact. Then for every club subset D of ω_1, there exist a club subset of $\gamma \in \omega_1$ such that $D \cap \gamma \nsubseteq^* C_\gamma$.*

Proof. Suppose that $S = \{\gamma \in \omega_1 \cap \mathrm{Lim} \colon D \cap \gamma \subseteq^* C_\gamma\}$ is stationary. Then for each $\gamma \in S$, there exists a $\zeta_\gamma < \gamma$ such that $(D \cap \gamma) \setminus \zeta_\gamma \subseteq C_\gamma$. By Fodor's lemma, there exist a stationary set $T \subseteq S$ and a single $\zeta < \omega_1$ such that for every $\gamma \in T$, $\zeta_\gamma = \zeta$. Let $E = \lim(D) \cap \omega_1$. Then E is τ-closed and $E \in \mathcal{F}$. Since (ω_1, τ) is perfectly normal, there exists a countable family $\{U_n \colon n < \omega\}$ of τ-open sets such that $\bigcap_{n<\omega} U_n = E$.

Fix $n \in \omega$. For every $\gamma \in U_n$, since U_n is τ-open, there exists an $\eta < \gamma$ such that $C_\gamma \setminus \eta \subseteq U_n$. If $\gamma > \zeta$, we may assume that $\eta \geqslant \zeta$. Then for every $\gamma \in (T \cap U_n) \setminus (\zeta + 1)$, there exists an $\eta < \gamma$ such that $(D \cap \gamma) \setminus \eta \subseteq C_\gamma \setminus \eta \subseteq U_n$. But since U_n contains a club subset E and T is stationary, $(T \cap U_n) \setminus (\zeta + 1)$ is stationary in ω_1. Thus there exist a stationary set $T_n \subseteq (T \cap U_n) \setminus (\zeta + 1)$ and an $\eta_n < \omega_1$ such that for every $\gamma \in T_n$, $(D \cap \gamma) \setminus \eta_n \subseteq U_n$. Therefore, we have $D \setminus \eta_n \subseteq U_n$.

Let $\eta_\omega = \sup_{n<\omega} \eta_n$. Then we have $D \setminus \eta_\omega \subseteq \bigcap_{n<\omega} U_n = E$. It clearly contradicts $E = \lim(D) \cap \omega_1$. $\square$

Now we are ready to prove the main result in this section.

Proposition 5.3. *Assume* PFA. *Let $\vec{C} = \langle C_\gamma \colon \gamma \in \omega_1 \cap \mathrm{Lim} \rangle$ be a guessing sequence on ω_1 so that C_γ is a closed subset of γ. Then $(\omega_1, \tau(\vec{C}))$ is not perfectly normal and non-realcompact.*

Proof. Suppose that $(\omega_1, \tau(\vec{C}))$ is perfectly normal and non-realcompact. Let $\tau = \tau(\vec{C})$. Then there exists a free z-ultrafilter $\mathcal{F}$ with countable intersection property. By Lemma 5.2, for every club subset D of ω_1, there exist a club subset of $\gamma \in \omega_1$ such that $D \cap \gamma \nsubseteq^* C_\gamma$.

Define a poset P by: $p = (s, X) \in P$ if and only if there exists an ordinal $\delta < \omega_1$ such that $s : \delta \to \omega$ and X is a subset of δ such that if γ is a limit point of X (γ may be δ), then for every $n < \omega$, there exists an $\zeta < \gamma$ such that $s(\xi) \geqslant n$ for every $\xi \in C_\gamma \setminus \zeta$. For each $q = (t, Y)$, $p = (s, X) \in P$, we let $q \leqslant p$ if and only if $t \supseteq s$, $Y \supseteq X$, and $Y \cap \mathrm{dom}(s) = X$.

Claim 1. *P is proper.*

Proof. Let λ be a sufficiently large regular cardinal and $\mathfrak{A} = \langle H(\lambda), \in, \triangle, \vec{C}, P\rangle$. In addition, let ν be a sufficiently large regular cardinal compared to λ and $\mathfrak{B} = \langle H(\nu), \in, \triangle, \mathfrak{A}\rangle$. Note that for every $X \subseteq H(\lambda)$, $Sk^{\mathfrak{A}}(X)$ is definable from X in $\mathfrak{B}$.

Now let N be a countable elementary substructure of $\mathfrak{B}$. We shall show that if $p \in N \cap P$, then there exists a $q \leqslant p$ which is (N, P)-generic. Let $\delta = N \cap \omega_1$. Let $\{\mathcal{D}_n\colon n < \omega\}$ be an enumeration of all open dense subsets lying in N. We shall construct a decreasing sequence $\langle p_n = (s_n, X_n)\colon n < \omega\rangle$ so that $p_n \in N$ and $p_{n+1} \in \mathcal{D}_n$ for every $n < \omega$. Moreover, we arrange that $s_{n+1}(\xi) = n$ for every $n < \omega$ and $\xi \in (\mathrm{dom}(s_{n+1}) \setminus \mathrm{dom}(s_n)) \cap C_\delta$.

Let $p_0 = p$. Suppose that we have defined p_n. Let $\delta_n = \mathrm{dom}(s_n)$. Since $Sk^{\mathfrak{A}}$ is definable in $\mathfrak{B}$, there exists an increasing continuous sequence $\langle N_i^n\colon i < \omega_1\rangle \in N$ such that $P, p_n, \mathcal{D}_n \in N_0^n$, $N_i^n \prec \mathfrak{A}$ and $\langle N_j^n\colon j \leqslant i\rangle \in N_{i+1}^n$ for every $i < \omega_1$. Let $D_n = \{N_i^n \cap \omega_1\colon i < \omega_1\}$. Then D_n is club and in N. By assumption, there exists a club subset E_n of ω_1 lying in N such that for every $\gamma \in E_n$, $D_n \cap \gamma \not\subseteq^* C_\gamma$. Hence we have $D_n \cap \delta \not\subseteq^* C_\delta$. Thus there exists a $\gamma \in D_n \cap (\delta_n, \delta)$ such that $\gamma \notin C_\delta$. By the definition of D_n, there exists an i such that $N_i^n \cap \omega_1 = \gamma$. Since $\gamma \notin C_\delta$ and C_δ is closed, γ is not a limit point of C_δ. Hence there exists a $\zeta < \gamma$ such that $C_\delta \cap [\zeta, \gamma) = \emptyset$. (Remark: this is the only place where we used C_δ is closed.) Let $p'_n = (s'_n, X'_n)$ be defined by: $\mathrm{dom}(s'_n) = \zeta + 1$, $s'_n \restriction \mathrm{dom}(s_n) = s_n$, $s'_n(\xi) = n$ for every $\xi \in (\mathrm{dom}(s'_n) \setminus \mathrm{dom}(s_n))$ and $X'_n = X_n$. Then we have $p'_n \in P \cap N_i^n$ and $p'_n \leqslant p_n$. Since $\mathcal{D}_n \in N_i^n$, there exists a $p_{n+1} \leqslant p'_n$ such that $p_{n+1} \in N_i^n \cap \mathcal{D}_n$. Then we have $(\mathrm{dom}(s_{n+1}) \setminus \mathrm{dom}(s_n)) \cap C_\delta \subseteq (\zeta + 1 \setminus \mathrm{dom}(s_n)) \cap C_\delta$. But by the definition of s'_n, for every $\xi \in (\zeta + 1) \setminus \mathrm{dom}(s_n)$, $s_{n+1}(\xi) = s'_n(\xi) = n$. Thus p_{n+1} satisfies the required condition.

Now let $q = (t, Y)$ be defined by: $t = \bigcup_{n<\omega} s_n$ and $Y = \bigcup_{n<\omega} X_n$. We claim $q \in P$. It suffices to show that for every $n < \omega$, there exists a $\zeta < \delta$ such that $t(\xi) \geqslant n$ for every $\xi \in C_\delta \setminus \zeta$. Fix $n < \omega$ and let $\zeta = \mathrm{dom}(s_n)$. Then by the construction of $\langle p_m\colon m < \omega\rangle$, if $n < m < \omega$, then $s_m(\xi) \geqslant n$ for every $\xi \in (\mathrm{dom}(s_m) \setminus \zeta) \cap C_\delta$. Thus we have $t(\xi) \geqslant n$ for every $\xi \in C_\delta \setminus \zeta$. It is easy to see that $q \leqslant p$ and q is (N, P)-generic. Hence P is proper. $\square$

For each $\gamma < \omega_1$, let $\mathcal{E}_\gamma = \{(s, X) \in P\colon \mathrm{dom}(s) > \gamma \text{ and } \sup(X) > \gamma\}$. Clearly $\mathcal{E}_\gamma$ is open dense. By PFA, we can take a $\{\mathcal{E}_\gamma\colon \gamma < \omega_1\}$-generic filter $G \subseteq P$. For each $n < \omega$, we define $D = \lim(\bigcup\{X\colon (s, X) \in G\}) \cap \omega_1$ and $F_n = \{\xi < \omega_1\colon s(\xi) = n \text{ for some } (s, X) \in G\}$ for each $n < \omega$. It is trivial that D is club, $\bigcup_{n<\omega} F_n = \omega_1$, and for every $\gamma \in D$, $F_n \cap C_\gamma$ is bounded in γ. For each $n < \omega$, let $\langle F_n^m\colon m < \omega\rangle$ be an increasing sequence so that $\bigcup_{m<\omega} F_n^m = F_n$ and for every $m < \omega$, every limit point of F_n^m belongs to D. Then it is easy to see that each F_n^m is τ-closed discrete. But since $\mathcal{F}$ has countable intersection property, there exist $m, n < \omega$ such that $F_n^m \in \mathcal{F}$. It is a contradiction. $\square$

Obviously, we may ask the following question.

Question 2. Does there always exist a guessing sequence $\vec{C}$ such that $(\omega_1, \tau(\vec{C}))$ is perfectly normal and non-realcompact?

This is an interesting special case of Blair's question.

6. *D*-spaces

The notion of D-spaces was introduced by van Douwen in [5]. For more information about this topological property, see [6,7]. It is defined as follows.

Definition 6.1. Let (X, τ) be a Hausdorff space. A τ-open neighborhood assignment (ONA) is a mapping $x \mapsto N_x$ with domain X such that each N_x is a τ-open neighborhood of x. If Y is a subset of X, we define $N(Y) = \bigcup_{x \in Y} N_x$.

X is said to be a D-space if and only if for every τ-ONA $x \mapsto N_x$, there exists a τ-closed discrete subset Y of X such that $N(Y) = X$.

Recall that when (X, τ) is a topological space,

$$\begin{aligned} e(X) &= \sup\bigl(\{\aleph_0\} \cup \bigl\{|Y|\colon\ Y \text{ is } \tau\text{-closed discrete subset of } X\bigr\}\bigr), \\ L(X) &= \sup\bigl(\{\aleph_0\} \\ &\qquad \cup \bigl\{\min\bigl\{|\mathcal{C}'|\colon\ \mathcal{C}' \text{ is a subcover of } \mathcal{C}\bigr\}\colon\ \mathcal{C} \text{ is an open covering of } X\bigr\}\bigr). \end{aligned}$$

It is easy to see that for every Hausdorff space (X, τ), if $e(Y) < L(Y)$ for some τ-closed subset Y of X, then (X, τ) is not a D-space. Since all of the known constructions of non-D-spaces rely on this easy observation, it was asked if whenever (X, τ) satisfies $e(Y) = L(Y)$ for every τ-closed subset Y of X, (X, τ) is a D-space. In this section, we shall solve this conjecture negatively. More precisely, the following theorem will be proved.

Theorem 6.2. *It is consistent that there exists a locally countable, first-countable, locally compact, regular, Hausdorff, non-D-space X such that for every closed subspace Y of X, $e(Y) = L(Y)$.*

We begin with a model of $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ and construct a countable-support iteration $\langle P_\alpha, \dot{Q}_\beta\colon\ \alpha \leqslant \omega_2,\ \beta < \omega_2\rangle$. In the course of construction, we define a P_{ω_2}-name $\dot{I}$ for a subset of ω_2 such that $\dot{I} \cap (\alpha + 1)$ is essentially a P_α-name for every $\alpha < \omega_2$. I denotes the interpretation of $\dot{I}$. We assume that P_α for every $\alpha \leqslant \omega_2$ is proper and adds no new real.

Let Q_0 be defined by: $q \in Q_0$ if and only if for some $\delta \in \omega_1 \cap \mathrm{Lim}$, q is a function with domain $(\delta + 1) \cap \mathrm{Lim}$ such that for every $\gamma \in (\delta + 1) \cap \mathrm{Lim}$,

(i) $q(\gamma)$ is an unbounded subset of γ,
(ii) $q(\gamma)$ is closed in the order topology in γ,
(iii) if $\xi \in q(\gamma) \cap \mathrm{Lim}$, then $q(\xi) =^* q(\gamma)$.

Q_0 is ordered by extension. If $G' \subseteq Q_0$ is generic, then for each $\gamma \in \omega_1 \cap \mathrm{Lim}$, we define $C_\gamma = q(\gamma)$ for some $q \in G'$ with $\gamma \in \mathrm{dom}(q)$, and set $\vec{C} = \langle C_\gamma\colon\ \gamma \in \omega_1 \cap \mathrm{Lim}\rangle$, and $\tau = \tau(\vec{C})$. By Lemmas 1.5, 2.1, and 2.3, (ω_1, τ) is a locally countable, first-countable, locally compact, regular, Hausdorff space. Let $\dot{C}_\gamma$ be a P_1-name for C_γ for every $\gamma \in \omega_1 \cap \mathrm{Lim}$, and $\dot{\tau}$ a P_1-name for τ.

Let $G_1 \subseteq P_1$ be generic. In $V[G_1]$, let Q_1 be the set of all q such that for some $\delta < \omega_1$, q is a function with domain $\delta + 1$ such that for every $\gamma \in \mathrm{dom}(q)$, $q(\gamma)$ is a τ-open neighborhood of γ with $q(\gamma) \subseteq \gamma + 1$. If $G' \subseteq Q_1$ is generic over $V[G_1]$, define $N_\gamma = q(\gamma)$ for some (all) $q \in G'$ such that $\gamma \in \mathrm{dom}(q)$. Clearly $\gamma \mapsto N_\gamma$ is a τ-ONA. This ONA will witness that (ω_1, τ) is not a D-space in the final model. Let $\dot{Q}_1$ be a P_1-name for Q_1 and for every $\gamma \in \omega_1 \cap \mathrm{Lim}$, $\dot{N}_\gamma$ a P_2-name for N_γ. We also let $0, 1 \notin I$.

Let $\langle \dot{F}_\alpha \colon 2 \leqslant \alpha < \omega_2 \rangle$ be a bookkeeping of all unbounded $\dot{\tau}$-closed subsets of ω_1. In an appropriate extension, F_α denotes the interpretations of $\dot{F}_\alpha$. While we are defining the iteration, we shall also define $P_{\alpha+1}$-names $\dot{D}_\alpha$ and $\dot{Y}_\alpha$ such that $\dot{D}_\alpha$ is forced to be a club subset of ω_1 and $\dot{Y}_\alpha$ is forced to be a $\dot{\tau}$-closed discrete subset of $\dot{F}_\alpha$.

Suppose that P_α has been defined for some $2 \leqslant \alpha < \omega_2$. Let $G_\alpha \subseteq P_\alpha$ be generic and work in $V[G_\alpha]$. If for some finite subset B of $I \cap \alpha$, $F_\alpha \setminus \bigcup_{\beta \in B} \dot{Y}_\beta$ is bounded in ω_1, then we let Q_α be the trivial poset and $\alpha \notin I$. Suppose for every finite subset B of $I \cap \alpha$, $\dot{F}_\alpha \setminus \bigcup_{\beta \in B} \dot{Y}_\beta$ is unbounded in ω_1. Let $\alpha \in I$. Define Q_α to be the set of all pairs $\langle s, t \rangle$ such that

(i) s is a closed bounded subset of ω_1,
(ii) t is a τ-closed discrete subset of $F_\alpha \cap (\max s + 1)$, and
(iii) for every $\xi \in t$, $N_\xi \cap s = \emptyset$.

For $q \in Q_\alpha$, let $\langle s^q, t^q \rangle = q$. For every $q, q' \in Q_\alpha$, let $q \leqslant q'$ if s^q is an end-extension of $s^{q'}$ and $t^q \cap (\max s^{q'} + 1) = t^{q'}$. Let $\dot{Q}_\alpha$ be a P_α-name for Q_α. When $G' \subseteq Q_\alpha$ is generic over $V[G_\alpha]$, we define $D_\alpha = \bigcup_{p \in G'} s^p$ and $Y_\alpha = \bigcup_{p \in G'} t^p$. It is easy to see that Y_α is a τ-closed discrete subset of F_α, and $N(Y_\alpha) \cap D_\alpha = \emptyset$. If $\alpha \notin I$, let $Y_\alpha = \emptyset$ and $D_\alpha = \omega_1$. In either case, it is clear that $N(Y_\alpha) \cap D_\alpha = \emptyset$. Let $\dot{D}_\alpha$ and $\dot{Y}_\alpha$ be $P_{\alpha+1}$-names for D_α and Y_α respectively.

The following lemma is the key for the proof of Theorem 6.2.

Lemma 6.3. *Let $\alpha \in [2, \omega_2)$. Then the following hold*:

(i) *Let M be a countable elementary substructure of $\langle H(\lambda), \in, \triangle \rangle$ with $P_\alpha \in M$ for some sufficiently large regular cardinal λ, $\delta = M \cap \omega_1$, $p \in P_\alpha \cap M$, and $\zeta < \delta$. Suppose that $\{\dot{X}_n \colon n < \omega\} \subseteq M$ is a countable set of P_α-names for subsets of ω_1 such that for every $n < \omega$, $p \Vdash$ 'for every finite subset B of α, $\dot{X}_n \setminus \bigcup_{\beta \in B} \dot{Y}_\beta$ is unbounded in ω_1'. Then there exists a $q \leqslant p$ such that q is totally (M, P_α)-generic, $q \restriction 1 \Vdash$ '$q(1)(\delta) \cap \zeta = \emptyset$', $q \Vdash$ '$\delta \notin \dot{Y}_\beta$ for every $\beta \in M \cap \alpha$', and for every $n < \omega$, $q \Vdash$ 'δ is a $\dot{\tau}$-limit point of $\dot{X}_n$'. In particular, P_α is totally proper.*
(ii) *If $p \Vdash$ '$\alpha \in \dot{I}$' for some $p \in P_\alpha$, then $p \Vdash$ 'for every $\zeta < \omega_1$, $\{q \in \dot{Q}_\alpha \colon \max t^q > \zeta\}$ is dense'. In particular, in $V^{P_{\alpha+1}}$, D_α is a club subset of ω_1.*

Proof. We go by induction.

Claim 1. *If both* (i) *and* (ii) *hold for every $\beta < \alpha$, then* (i) *holds for α.*

Proof. Let M, λ, p, δ, ζ, and $\{\dot{X}_n\colon n<\omega\}$ be as in the assumption of (i). Without loss of generality, we may assume that $\mathrm{dom}(p(0)) \subseteq \zeta$, and $p \restriction 1 \Vdash$ '$\mathrm{dom}(p(1)) \subseteq \zeta$'.

Let $\langle p_i\colon i<\omega\rangle$ be a (M, P_α)-generic sequence with $p_0 = p$. Let $\bar{I} = \{\beta \in \alpha \cap M\colon \exists i < \omega(p_i \Vdash \text{'}\beta \in \dot{I}\text{'})\}$, $\langle \delta_n\colon n<\omega\rangle$ an increasing cofinal sequence in δ with $\delta_0 \geqslant \zeta$, and $\langle k_n\colon n<\omega\rangle$ a sequence such that for every $k<\omega$, there are infinitely many $n<\omega$ such that $k = k_n$. We shall build an increasing sequence $\langle \gamma_n\colon n<\omega\rangle$ in δ lying in M and a sequence $\langle W_n\colon n<\omega\rangle$ as follows.

Suppose that we have defined γ_m and W_m for every $m<n$. Let $\delta'_n = \max\{\delta_n, \sup_{m<n}\gamma_m\}$. By assumption, $p_i \Vdash$ '$\dot{X}_{k_n} \setminus \bigcup_{m\leqslant n} \dot{Y}_{\beta_m}$ is unbounded' for every $i<\omega$. Since $\langle p_i\colon i<\omega\rangle$ is a (M, P_α)-generic sequence, there exists a $\gamma_n \in (\delta'_n, \delta)$ such that $p_i \Vdash$ '$\gamma_n \in \dot{X}_{k_n} \setminus \bigcup_{m\leqslant n} \dot{Y}_{\beta_m}$' for some $i<\omega$. If γ_n is a successor ordinal, let $W_n = \{\gamma_n\}$. Otherwise, since $p \Vdash$ '$\bigcup_{m\leqslant n} \dot{Y}_{\beta_m}$ is $\dot{\tau}$-closed discrete', there exists a successor ordinal $\zeta_n \in (\delta'_n, \gamma_n)$ such that $p_i \Vdash$ '$(\dot{C}_{\gamma_n} \setminus \zeta_n) \cap \bigcup_{m\leqslant n} \dot{Y}_{\beta_m} = \emptyset$' for some $i<\omega$. Let W_n be such that for some $i<\omega$, $p_i \Vdash$ '$W_n = \dot{C}_{\gamma_n} \setminus \zeta_n$'. Without loss of generality, we may assume $\zeta_n \geqslant \delta'_n$.

Define $p' \in P_\alpha$ by induction. First let

$$p'(0) \restriction \delta = \bigcup_{i<\omega} p_i(0),$$

$$p'(0)(\delta) = \bigcup_{n<\omega} W_n,$$

$$p' \restriction 1 \Vdash \text{'}p'(1) = \bigcup_{i<\omega} p_i(1) \cup \bigl\{\bigl\langle \delta, (\zeta, \delta)\bigr\rangle\bigr\}\text{'}.$$

As in the previous arguments, we can show that $p' \restriction 2 \in P_2$.

We will make sure $\mathrm{supp}(p') = \{0,1\} \cup \bar{I}$. Since $\mathrm{supp}(p')$ is countable, we have no problem at limit stages. Suppose that $p' \restriction \beta'$ has been defined for $\beta' \leqslant \beta$. If $\beta \notin \bar{I}$, let $\beta \notin \mathrm{supp}(p')$. If $\beta \in \bar{I}$, we let $s^{p'(\beta)}$ and $t^{p'(\beta)}$ be P_β-names such that $p' \restriction \beta$ forces

$$s^{p'(\beta)} = \bigcup_{i<\omega} s^{p_i(\beta)} \cup \{\delta\},$$

$$t^{p'(\beta)} = \bigcup_{i<\omega} t^{p_i(\beta)}.$$

We claim that $p' \restriction (\beta+1) \in P_{\beta+1}$. It suffices to show that $p' \restriction \beta \Vdash$ '$\dot{C}_\delta \cap t^{p'(\beta)}$ is bounded in δ'. Since $\beta \in \bar{I}$, there exists an $m<\omega$ such that $\beta = \beta_m$. Note that $p' \restriction \beta \Vdash$ '$t^{p'(\beta)} = \dot{Y}_\beta \cap \delta$'. Then by construction, for every $n \geqslant m$, $p' \restriction \beta \Vdash$ '$W_n \cap t^{p'(\beta)} = W_n \cap \dot{Y}_\beta = \emptyset$'. Therefore, $p' \restriction \beta \Vdash$ '$(\dot{C}_\delta \setminus \delta'_m) \cap t^{p'(\beta)} = (\bigcup_{m\leqslant n<\omega} W_n) \cap t^{p'(\beta)} = \emptyset$'. Thus $p' \restriction (\beta+1) \in P_{\beta+1}$. It completes the proof of $p' \in P_\alpha$.

Since it is a lower bound of an (M, P_α)-generic sequence, p' is totally (M, P_α)-generic. By construction, $p' \restriction 1 \Vdash$ '$p'(1)(\delta) \cap \zeta = \emptyset$' and for every $\beta < \alpha$, $p' \restriction \beta \Vdash$ '$\delta \notin \dot{Y}_\beta$'.

Now it suffices to show that for every $k<\omega$, $p' \Vdash$ 'δ is a $\dot{\tau}$-limit point of $\dot{X}_k$'. Let $\varepsilon < \delta$. Then there exists an $n<\omega$ such that $k = k_n$ and $\delta_n \geqslant \varepsilon$. Then $p' \Vdash$ '$\varepsilon < \gamma_n \in \dot{X}_k \cap \dot{C}_\delta$'. It follows that $p' \Vdash$ 'δ is a $\dot{\tau}$-limit point of $\dot{X}_k$'. $\square$

Claim 2. *If* (i) *holds for* α, *then* (ii) *holds for* α.

Proof. Let $p_0 \in P_\alpha$ with $p_0 \Vdash$ '$\alpha \in \dot{I}$' and $\dot{q}_0$ a P_α-name such that $p_0 \Vdash$ '$\dot{q}_0 \in \dot{Q}_\alpha$'. Also let $\zeta < \omega_1$. There exist $p_1 \leqslant p_0$ and $\zeta' < \omega_1$ such that $p_1 \Vdash$ '$\max s^{\dot{q}_0} = \zeta'$'. Let $\gamma = \max\{\zeta, \zeta'\}$. Since $p_0 \Vdash$ '$\alpha \in \dot{I}$', we have $p_1 \Vdash$ 'for every finite subset B of $\dot{I} \cap \alpha$, $\dot{F}_\alpha \setminus \bigcup_{\beta \in B} \dot{Y}_\beta$ is unbounded in ω_1'. Let M be a countable elementary substructure of $\langle H(\lambda), \in, \triangle\rangle$ with $P_\alpha, \dot{Q}_\alpha, \dot{F}_\alpha, p_1, \dot{q}_0, \gamma \in M$ and $\delta = M \cap \omega_1$. By (i) for α, there exists an (M, P_α)-generic condition $p_2 \leqslant p_1$ such that $p_2 \restriction 1 \Vdash$ '$p_2(1)(\delta) \cap (\gamma + 1) = \emptyset$' and $p_2 \Vdash$ 'δ is a $\dot{\tau}$-limit point of $\dot{F}_\alpha$'. Let $\dot{q}_1$ be a P_α-name such that $p_2 \Vdash$ '$s^{\dot{q}_1} = s^{\dot{q}_0} \cup \{\delta + 1\}$ and $t^{\dot{q}_1} = t^{\dot{q}_0} \cup \{\delta\}$'. Then $p_2 \Vdash$'$\dot{N}_\delta \cap s^{\dot{q}_1} = p_2(1)(\delta) \cap \dot{s}^{\dot{q}_1} \subseteq (\zeta, \delta] \cap (\zeta \cup \{\delta + 1\}) = \emptyset$'. It follows that $p_2 \Vdash$ '$\dot{q}_1 \in \dot{Q}_\alpha$'. Therefore, $p_0 \Vdash$ 'for every $\zeta < \omega_1$, $\{q \in \dot{Q}_\alpha\colon \max t^q > \zeta\}$ is dense'. □

Let $P = P_{\omega_2}$. By the previous lemma, it is easy to see that all unbounded τ-closed discrete sets in V^P are explicitly added by the forcing.

Lemma 6.4. *In* V^P, *for every* τ*-closed discrete set* F, *there exists a finite subset* B *of* I *such that* $F \setminus \bigcup_{\beta \in B} Y_\beta$ *is countable.*

Proof. Let $p \in P$ and suppose that $\dot{F}$ is a P-name for a $\dot{\tau}$-closed subset of ω_1 such that $p \Vdash$ 'for every finite subset B of ω_2, $\dot{F} \setminus \bigcup_{\beta \in B} \dot{Y}_\beta$ is unbounded'. We shall show that there exists a $p' \leqslant p$ which forces that $\dot{F}$ is not $\dot{\tau}$-closed discrete. Let M be a countable elementary substructure of $\langle H(\lambda), \in, \triangle\rangle$ for some sufficiently large regular cardinal λ such that $P, p, \dot{r} \in M$. Set $\delta = M \cap \omega_1$. Then by Lemma 6.3, there exists a $p' \leqslant p$ such that $p' \Vdash$ 'δ is a $\dot{\tau}$-limit point of $\dot{F}$'. It follows that $p' \Vdash$ '$\dot{F}$ is not $\dot{\tau}$-closed discrete'. □

Given these lemmas, it is easy to show Theorem 6.2.

Proof of Theorem 6.2. Let $G \subseteq P$ be generic over V and work in $V[G]$. Let $\tau = \tau(\vec{C})$ and $\dot{\tau}$ a name for τ. First we shall show that $\gamma \mapsto N_\gamma$ witnesses that (ω_1, τ) is not a D-space. Let Y be a τ-closed discrete set. Then by Lemma 6.4, there exists a finite subset B of ω_2 such that $Y \setminus \bigcup_{\beta \in B} Y_\beta$ is bounded by some $\zeta < \omega_1$. Then we have $N(Y) \subseteq N(\bigcup_{\beta \in B} Y_\beta) \cup N(\zeta) \subseteq \bigcup_{\beta \in B} N(Y_\beta) \cup \zeta$. However, by construction, for each $\beta \in B$, $D_\beta \cap N(Y_\beta) = \emptyset$. Thus $\bigcap_{\beta \in B} D_\beta \cap \bigcup_{\beta \in B} N(Y_\beta) = \emptyset$. Since each D_β is a club subset of ω_1, $\bigcap_{\beta \in B} D_\beta$ is club. Thus we can conclude $N(Y) \neq \omega_1$.

Now it suffices to show that for every τ-closed set F, $e(F) = L(F)$. If F is countable, then clearly $e(F) = L(F) = \aleph_0$. Suppose that F is uncountable. Since (ω_1, τ) is locally countable, we have $L(F) = \aleph_1$. We shall show that there exists a τ-closed discrete subset of F of size $\aleph_1$. If there exists a finite subset B of ω_2 such that $F \setminus \bigcup_{\beta \in B} Y_\beta$ is countable, then $F \cap \bigcup_{\beta \in B} Y_\beta$ is unbounded in ω_1 and τ-closed discrete since it is a finite union of τ-closed discrete sets. Otherwise, by the construction of P, there exists an $\alpha < \omega_2$ such that $F = F_\alpha$. It follows that $Y_\alpha \subseteq F$ is a τ-closed discrete set. Thus in either way, there exists an unbounded τ-closed discrete subset of F. □

Because the proof is similar to the one in Section 3, one may wonder if by using the idea of Section 4, we can obtain the model of $\mathrm{MA} + \neg\mathrm{CH}$ which witnesses Theorem 6.2. However, we do not know if it can be done.

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